

## Review Article

# Spin Foam Models with Finite Groups

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Spin foam models, loop quantum gravity, and group field theory are discussed as quantum gravity candidate theories and usually involve a continuous Lie group. We advocate here to consider quantum gravity-inspired models with finite groups, firstly as a test bed for the full theory and secondly as a class of new lattice theories possibly featuring an analogue diffeomorphism symmetry. To make these notes accessible to readers outside the quantum gravity community, we provide an introduction to some essential concepts in the loop quantum gravity, spin foam, and group field theory approach and point out the many connections to the lattice field theory and the condensed-matter systems.

## 1. Introduction

Spin foam models [1–8] arose as one possible quantization method for gravity. These models can be seen in the tradition of lattice approaches to quantum gravity [9], in which gravity is formulated as a statistical system on either a fixed or fluctuating lattice. In a rather independent fashion, lattice methods are also ubiquitous in condensed-matter and (Yang-Mills) gauge theories. It is interesting to note that the lattice structures appearing in the strong coupling expansion of QCD or in the high-temperature expansion for the Ising gauge [10] systems are of spin foam type.

In any lattice approach to gravity, one has to discuss the significance of the (choice of) lattice for physical predictions. Although this point arises also for other lattice field theories, it carries extreme importance for general relativity. When it comes to matter fields on a lattice, we can interpret the lattice itself as providing the background geometry and therefore background space time. This could not contrast more with the situation in general relativity, where geometry, and therefore space time itself, is a dynamical variable and has to be an outcome rather than an ingredient of the theory. This is one aspect of background independence for quantum gravity [11, 12].

One possibility, to alleviate the dependence of physical results on the choice of lattice, is to turn the lattice itself

into a dynamical variable and to sum over (a certain class of) lattices. This is one motivation for the dynamical triangulation approach [13–15], causal dynamical triangulations [16, 17], and tensor model/tensor group field theory [18–26]. See also [27, 28] for another possibility to introduce a dynamical lattice. Once one decides to sum over lattice structure, one must provide a prescription to do so. The class of triangulations to be summed over can be restricted, for instance, in order to implement causality [16, 17, 29] or to symmetry—reduce models [30–32].<sup>1</sup>

Another possibility is to ask for models which are *per se* lattice or discretization independent. Such models might arise as fixed points of a Wilsonian renormalization flow and represent the so-called perfect discretizations [33–38]. For gravity, this concept is intertwined with the appearance of discrete representation of diffeomorphism symmetry in the lattice models [36–41]. Such a discrete notion of diffeomorphism symmetry can arise in the form of vertex translations [42–46]. Discrete geometries can be represented by a triangulation carrying geometric data, for instance, in the Regge calculus [47, 48], the lengths of the edges in the triangulation. Vertex translations are then symmetries that act (locally) on the vertices of the triangulations by changing the geometric data of the adjacent building blocks. If the symmetry is fully implemented into the model, then this action should not change the weights in the partition

function. Such vertex translation symmetries are realized in 3D gravity, where general relativity is a topological theory [49]. This means that the theory has no local degrees of freedom, only a topology-dependent finite number of global ones. The first-order formalism of 3D gravity coincides with a 3D version of the *BF*-theory [50]. The *BF*-theory is a gauge theory and can be formulated with a gauge group given by a Lie group. The 3D gravity is obtained by choosing  $SU(2)$ . However, one can also choose finite groups [51–53]. In this case, one obtains models used in quantum computing [54] as well as models describing topological phases of condensed-matter systems, so called string-net models [55]. Moreover the *BF*-systems can be seen as Yang-Mills-like systems for a special choice of the coupling parameter (corresponding to zero temperature). Hence, in 3D, we have Yang-Mills-like theories with the usual (lattice) gauge symmetry as well as the topological *BF*-theories with the additional translation symmetries.

The *BF*-theory (in any dimension) is a topological theory because it has a large gauge group of symmetries, which reduces the physical degrees of freedom to a finite number. It is characterized not only by the usual (lattice) gauge symmetry determined by the gauge group but also by the so-called translational symmetries, based on the 3-cells (i.e., cubes) of the lattice and parametrized by the Lie algebra elements (if we work with the Lie groups). For 3D gravity, these symmetries can be interpreted as being associated to the vertices of the dual lattice (for instance, given by a triangulation) and are hence termed vertex translations. However, for 4D, the symmetries are still associated to the 3-cells of the lattice and hence to the edges of the dual lattice or triangulation. As in general there are more edges than vertices in a triangulation, we obtain much more symmetries than the vertex translations one might look for in 4D gravity. To obtain vertex translations of the dual lattice in 4D, one would rather need a symmetry based on the 4-cells of the lattice.

In 4D, gravity is not a topological theory. There is however a formulation due to Plebański [56] that starts with the *BF*-theory and imposes the so-called simplicity constraints on the variables. These break the symmetries of the *BF*-theory down to a subgroup that can be interpreted as diffeomorphism symmetry (in the continuum). This Plebanski formulation is also the one used in many spin foam models. Here, one of the main problems is to implement the simplicity constraints [57–64] into the *BF*-partition function. The status of the symmetries is quite unclear in these models; however, there are indications [40] that in the discrete the reduction of the *BF*-translation symmetries does not leave a sufficiently large group to be interpretable as diffeomorphisms. Nevertheless, if these models are related to gravity, then there is a possibility that diffeomorphism symmetry as the fundamental symmetry of general relativity arises as a symmetry for large scales. In the abovementioned method of regaining symmetries by coarse graining, one can then expect that this diffeomorphism symmetry arises as vertex translation symmetry (on the vertices of the dual lattice).

Hence, we want to emphasize that, in 4D, spin foam models are candidates for a new class of lattice models in addition to Yang-Mills-like systems and the *BF*-theory. This

new class would be characterized by a symmetry group in between those for the Yang-Mills theory (usual lattice gauge symmetry) and *BF*-symmetry (usual lattice gauge symmetry and translation symmetry associated to the 3-cells). Here, we would look for a symmetry given by the usual lattice gauge symmetry and translation symmetries associated to the 4-cells.

As we will show in this work, the 4D spin foam models, which as gravity models are based on  $SO(4)$ ,  $SO(3,1)$ , or  $SU(2)$ , can be easily generalized to finite groups (or more generally tensor categories [65, 66]). Although the immediate interpretation as gravity models is lost, one can nevertheless ask whether translation symmetries are realized, and if not, how these could be implemented. This question is much easier to answer for finite groups than in the full gravity case. One can therefore see these models as a test bed for the full theory. This also applies for renormalization and coarse graining techniques which need to be developed to access the large-scale limit of spin foams. With finite group models, it might be in particular possible to access the many-particle (that is many simplices or building blocks in the triangulation) and small-spin (corresponding to small geometrical size of the building blocks) regime.<sup>2</sup> This is in contrast to the few-particle and large-spin (semiclassical) regime [67–69] which is accessible so far.

In the emergent gravity approaches [70–72], one attempts to construct models, which do not necessarily start from gravitational or even geometrical variables but nevertheless show features typical of gravity. The models proposed here can also be seen as candidates for an emergent gravity scenario. A related proposal is the truncated Regge models in [73–76], in which the continuous length variables of the Regge calculus are replaced by values in  $\mathbb{Z}_q$ , with  $q = 2$  for “the Ising quantum gravity”.

Apart from providing a wealth of test models for quantum gravity researchers, another intention of this work is to give an introduction to some of the spin foam concepts and ideas to researchers outside quantum gravity. Indeed, spin foams arise as graphical tools in high-temperature expansions of lattice theories, or more generally in the construction of dual models. We will review these ideas in Section 2 where we will discuss the Ising-like systems and introduce spin nets as a graphical tool for the high-temperature expansion. Next, in Section 3, we will discuss lattice gauge theories with the “Abelian” finite groups. Here, spin foams arise in the high-temperature expansion. The zero-temperature limit of these theories gives topological *BF*-theories which we will discuss in more detail in Section 4. In Section 5, we detail spin foam models with the non-Abelian finite groups. Following a strategy from quantum gravity, we will also discuss the so-called constrained models which arise from the *BF*-models by implementing simplicity constraints, here in the form of edge projectors, into the partition function. This will in general change the topological character of the *BF*-models to nontopological ones.

We will then discuss in Section 6 a canonical description for lattice gauge theories and show that for the *BF*-theories the transfer operators are given by projectors onto the so-called physical Hilbert space. These are known as stabilizer

conditions in quantum computing and in the description of string nets. Here, we will discuss the possibility to alter the projectors in order to break some subset of the translation symmetries of the  $BF$ -theories.

Next, in Section 7, we give an overview of group field theories for finite groups. Group field theories are quantum field theories on these finite groups and generate spin foams as the Feynman diagrams. Finally, in Section 8, we discuss the possibility of nonlocal spin foams. We will end with an outlook in Section 9.

## 2. The Ising-Like Models

We will review the construction of the models [77, 78] that are dual to the Ising-like models. Henceforth, they will be referred to as *dual models*. As we will see, similar techniques lead to the spin foam representation for (Ising-like) gauge theories. The main ingredient comprises of a Fourier expansion of the couplings, which can be understood as functions of group variables. We will concentrate on the groups  $\mathbb{Z}_q$ , with  $q = 2$  for the Ising models proper.

**2.1. Spin Foam Representation.** Consider a generic lattice, of regular or random type,<sup>3</sup> in which the edges are oriented. (In principle, this last condition is not necessary for the Ising models; that is,  $q = 2$ ). One defines an Ising model on such a lattice, with spins  $g_v$  associated to the vertices and nearest-neighbour interactions, by utilizing the following partition function:

$$Z = \frac{1}{2^{\#v}} \sum_{g_v} \prod_e \exp(\beta g_{s(e)} g_{t(e)}^{-1}), \quad (1)$$

where  $g_v \in \{0, 1\}$ ,  $\beta$  is a coupling constant, which is inversely proportional to temperature, and  $\#v$  is the number of vertices in the lattice. Note that the group product is the standard product on  $\mathbb{Z}_2$ , that is, addition modulo 2.<sup>4</sup> The action  $S = \beta \sum_e g_{s(e)} g_{t(e)}^{-1}$  describes the coupling between the spin  $g_{s(e)}$  at the source or starting vertex and the spin  $g_{t(e)}$  at the target or terminating vertex of every edge.

More generally, one can assume that  $g_v \in \{0, \dots, q-1\}$  (giving the various vector Potts models). The group product is the standard one on  $\mathbb{Z}_q$ , that is, addition modulo  $q$ . A further generalization of (1) is to allow the edge weights  $w_e$  to be (even locally varying) functions of the two group elements ( $g_{s(e)}$  and  $g_{t(e)}$ ) associated to the edge  $e$ :

$$Z = \frac{1}{q^{\#v}} \sum_{g_v} \prod_e w_e(g_{s(e)} g_{t(e)}^{-1}). \quad (2)$$

Now, these group-valued functions  $w_e$  can be expanded with respect to a basis given by the irreducible representations of the group [79]. For the groups  $\mathbb{Z}_q$ , this is just the usual discrete Fourier transform:

$$\begin{aligned} w(g) &= \sum_{k=0}^{q-1} \tilde{w}(k) \chi_k(g), \\ \tilde{w}(k) &= \frac{1}{q} \sum_{g=0}^{q-1} w(g) \bar{\chi}_k(g), \end{aligned} \quad (3)$$

where  $\chi_k(g) = \exp((2\pi i/q)k \cdot g)$  are the group characters. Characters for the Abelian groups are multiplicative; that is,  $\chi_k(g_1 \cdot g_2) = \chi_k(g_1) \cdot \chi_k(g_2)$  and  $\chi_k(g^{-1}) = [\chi_k(g)]^{-1} = \bar{\chi}_k(g)$ .

Applying the character expansion to the weights in (2), one obtains (using the multiplicative property of the characters)

$$\begin{aligned} Z &= \frac{1}{q^{\#v}} \sum_{g_v} \sum_{k_e} \left( \prod_e \tilde{w}_e(k_e) \chi_{k_e}(g_{s(e)} g_{t(e)}^{-1}) \right) \\ &= \frac{1}{q^{\#v}} \sum_{g_v} \sum_{k_e} \left( \prod_e \tilde{w}_e(k_e) \right) \left( \prod_e \chi_{k_e}(g_{s(e)} g_{t(e)}^{-1}) \right) \\ &= \frac{1}{q^{\#v}} \sum_{k_e} \left( \prod_e \tilde{w}_e(k_e) \right) \prod_v \left( \sum_{g_v} \prod_{e \supset v} \chi_{k_e}(g_v^{o(v,e)}) \right), \end{aligned} \quad (4)$$

where  $o(v, e) = 1$  if  $v$  is the source of  $e$  and  $-1$  if  $v$  is the target of  $e$ . Here, we simply resorted the product so that in the last line the factor  $\prod_{e \supset v} \chi_{k_e}(g_v^{o(v,e)})$  involves all of the appearances of  $g_v$ . We can now sum over  $g_v$ , which according to the Fourier inversion theorem gives a delta function involving the  $k_e$  of the edges adjacent to  $v$ :

$$Z = \sum_{k_e} \left( \prod_e \tilde{w}_e(k_e) \right) \prod_v \delta^{(q)} \left( \sum_{e \supset v} o(v, e) k_e \right), \quad (5)$$

where  $\delta^{(q)}(\cdot)$  is the  $q$ -periodic delta function.<sup>5</sup>

We have represented the partition function in a form where the group variables are replaced by variables taking values in the set of representation labels. This set inherits its own group product via this transform and is known as the Pontryagin dual of  $G$ . For  $\mathbb{Z}_q$ , the Pontryagin dual and the group itself are isomorphic, as both  $g, k \in \{0, \dots, q-1\}$ . We will call a representation of the form of (5), where the sum over group elements is replaced by a sum over representation labels, a *spin foam representation*.<sup>6</sup> Generically speaking, representations meeting at vertices must satisfy a certain condition. For the Abelian models above, this condition stated that the oriented sum of the representation labels, meeting at a given vertex, had to vanish. In other words, the tensor product of the representations meeting at an edge has to be trivial. In other models, one finds other characteristics as the following. (i) For the non-Abelian groups, this triviality condition generalizes to the choice of a projector from the tensor product of all representations meeting at a vertex into the trivial representation. This choice is encoded as additional information attached to the vertices. (ii) For the gauge theory models we will examine later, the representations are attached to faces, while the triviality condition involves faces sharing a given edge.

Note that the number of representation labels  $k_e$  differs from the number of group variables  $g_v$ , since the number of edges is generally larger than the number of vertices. In addition, however, these representation labels are subject to constraints. These are referred to as the Gauss constraints, since they demand that the analogue of the electric field (the representation labels) be divergence free (the sum of

the representation labels meeting at a vertex has to be zero modulo  $q$ ).

**2.2. Dual Models.** To finalize the construction of the dual models [78], one solves the Gauss constraints and replaces the representation labels associated to the edges of the lattice by variables defined on the dual lattice.

- (2d) A particularly clear example arises in two dimensions. On a differentiable manifold, a divergence-free vector field  $\vec{v}$  can be constructed from a scalar field  $\phi$  via  $v_1 = \partial_2 \phi$ ,  $v_2 = -\partial_1 \phi$ . On the lattice, a similar construction leads to a dual model with variables associated to the vertices of the dual lattice.<sup>7</sup> Indeed, for  $\mathbb{Z}_2$ , one finds again the Ising model, just that the high- (low-) temperature regime of the original model is mapped to the low- (high-) temperature regime of the dual model.
- (3d) In three dimensions, a divergence-free vector field  $\vec{v}$  can be constructed from another vector field  $\vec{w}$  by taking its curl:  $v_i = \epsilon_{ijk} \partial_j w_k$ . However, adding a gradient of a scalar field to  $\vec{w}$  does not change  $\vec{v}$ :  $\vec{w} \rightarrow \vec{w} + \vec{\partial} \phi$  leaves  $\vec{v} \rightarrow \vec{v}$ . This translates into a local gauge symmetry for the dual lattice theory. For the dual lattice model, variables reside on the edges of the dual lattice, and the weights define couplings among the edges bounding the faces (that is, the two-dimensional cells) of the dual lattice. Moreover, the aforementioned local gauge symmetry is realized at the vertices of the dual lattice (since the continuum gauge symmetry is parametrized by a scalar field). As a result, the dual model is a gauge theory, a class of theories that we will discuss in more detail in Section 3.
- (4d) In four dimensions, a similar argument to that just made in three dimensions leads to a class of dual models that are “higher-gauge theories” [80, 81]: the variables are associated to the faces of the dual lattice, and the weights define couplings among the faces bounding three-dimensional cells.
- (1d) Finally, let us examine one-dimensional models. Every vertex is adjacent to two edges. Hence, the Gauss constraints, expressed in (5), force the representation labels to be equal (we assume periodic boundary conditions). We can easily find that

$$Z = q^{\sum_v} \sum_k \left( \prod_e \tilde{w}_e(k) \right) = q^{\sum_v} \sum_k \tilde{w}(k)^{\sum_v e}. \quad (6)$$

For the second equality above, we are restricted to couplings that are homogenous on the lattice; that is, they do not depend on the position or orientation of the edge.

**2.3. High-Temperature Expansion and Spin Nets.** This spin foam representation is well adapted to describe the high-temperature expansion [82]. Indeed, if one considers the Ising

model with couplings as in (1), one obtains the following coefficients in the character expansion:

$$\begin{aligned} \tilde{w}(0) &= e^{\beta/2} \cosh \beta, \\ \tilde{w}(1) &= e^{\beta/2} \cosh \beta \tanh \beta. \end{aligned} \quad (7)$$

One can now expand the partition function (5) in powers of the ratio  $\tilde{w}(1)/\tilde{w}(0)$ , that is, into terms distinguished by the number of times the representation labelled  $k_e = 1$  appears.

The “ground” state is the configuration with  $k_e = 0$  for all edges. Edges with  $k_e = 1$  are said to be “excited”. The Gauss constraints in (5) enforce that the number of “excited” edges incident at each vertex must be even. On a cubical lattice, therefore, the lowest-order contribution arises at  $(\tilde{w}(1)/\tilde{w}(0))^4$  from those configurations with excited edges on the boundary of a single plaquette. The full amplitude at this order involves a combinatorial factor, namely, the number of ways one can embed the square (with four edges) into the lattice.

This reasoning extends to all Ising-like models and to higher-order terms in the perturbation series, which can be represented by closed graphs, known as polymers, embedded into the lattice. The expansion in terms of these graphs can then be organized in different ways [82]. Consider one particular contribution to the expansion, that is, a configuration where a some subset of edges carries nontrivial representation labels. In general, one may decompose such a subset into connected components, where one defines two sets of edges as disconnected if these sets do not share a vertex.

We will call such a connected component a *restricted spin net*, where restricted refers to the condition that all edges must carry a nontrivial representation. The Kronecker- $\delta$ s in (5) forbid the spin net to have open ends; that is, all vertices are bivalent or higher. (For the Ising model, with its lone nontrivial representation, only even valency is allowed.) Furthermore, edges meeting at a bivalent vertex, whether there is a “change of direction” or not, must carry the same representation label. (On this point, we have assumed that the couplings  $w_e$  depend neither on the positions nor on the directions of the edges; otherwise, one must equip the spin nets with more embedding data.) The representation labels may only change at vertices that are trivalent or higher, so called branching points. This allows one to define *geometric (restricted) spin nets*,<sup>8</sup> which are specified by two quantities: (i) the *connectivity* of the spin net vertices, that is, the valency of the branching points within the spin net and (ii) the *length* of the edges of the spin net, that is, the number of lattice edges that make up a spin net edge. The contribution of such a spin net to the free energy<sup>9</sup> can be split into two parts, one is specific to the spin net itself and, from (4), one records that it is given by

$$\prod_{e: k_e \neq 0} \tilde{w}_e(k_e). \quad (8)$$

The other part is a combinatorial factor determined by the number of embeddings of the spin net into the ambient lattice. This could be summarized as a measure (or entropic) factor, which carries information about the lattice in question, including its dimension.



**2.4. Some Considerations for Quantum Gravity.** To reiterate, one notes from (8) that the spin net-specific contribution factorizes with respect to the edges of the spin net; for an edge with representation  $k$ , it is given by  $\bar{w}(k)^l$  where  $l$  is the length of the spin net edge. Meanwhile, the combinatorial factors are determined by the ambient lattice and its structure. To describe quantum gravity, one is interested rather more in “background-independent” models, where such factors should not play a role. In this context, one can define an alternative partition function over geometric spin nets, whose edges carry both a representation label and an edge-length label. This length variable encodes the geometry of space time as a dynamical variable on the geometric spin net.

From that point of view, “background independence” motivates the search for models, within which the spin net weights are independent of these edge lengths. These can be termed *abstract spin nets* [83]. (Note that, with this definition, the abstract spin nets generally still remember some features of the ambient lattice; for instance, the valency of the spin net cannot be higher than the valency of the lattice.) For the Ising model, such abstract spin nets arise in the limit  $\tanh \beta = 1$ , that is, for zero temperature. In contrast to the high-temperature expansion, which is a sum ordered according to the length of the excited edges, abstract spin networks arise in a regime where this length does not play a role.

A similar structure arises in string-net models [55], which were designed to describe condensed phases of (scale-free) branching strings. Indeed, these string-net models are closely related to (the canonical formulation of) topological field theories, such as the *BF*-theories, which we will describe in Section 6. In this canonical formulation, spin net(work)s appear as one choice of basis for the Hilbert space of the *BF*-theories. String-net models assign amplitudes (corresponding to the so-called physical wave functions in the *BF*-theory) to spin networks that satisfy certain (gauge) symmetry properties. This implies that these spin networks can be freely deformed on the lattice without changing their amplitude.

In the following section, we will consider gauge systems, for which we will devise once again a spin foam representation. For these gauged models, rather than labelling the terms in the expansion using spin nets, we will utilize a different, yet similar structure known as a *spin foam*. These have a similar structure to spin nets, except that the various roles are played by simplices one dimension higher. To clarify, the roles of the vertices and edges are assumed by the edges and faces, respectively. With this proviso, other concepts introduced above translate nicely. Spin foam edges and faces are generally comprised of several edges and faces in the ambient lattice. Moreover, one can define both geometric and abstract spin foams [83]. Both are branched surfaces [1–6, 84] whose constituent faces carry a nontrivial representation label.<sup>10</sup> The amplitudes for geometric spin foams will most often depend on the area of their surfaces, that is, the number of plaquettes in the ambient lattice constituting the spin foam surface in question, as well as the lengths of their branches, that is, the number of edges in the ambient lattice constituting the spin

foam branch in question. For instance, this is the case in the Yang-Mills-like theories, where geometric spin foams arise in the strong coupling or high-temperature expansion.

The conditions for *abstract spin foams*, that is, amplitudes independent of the surface area, may be understood as requiring that the amplitudes be invariant under trivial (face and edge) subdivision. This can be used to specify measure factors for the spin foam models [1–5, 83, 85–87]. Abstract spin foams are also generated by the tensor models discussed in Section 7; however, the spin foams thus generated are not “restricted” in the manner we defined earlier; that is, the trivial representation label is allowed. Furthermore, the spin foams need not be embeddable into a given lattice. See also [88] for one possible connection between partition functions with restricted and unrestricted spin foams.

This discussion on geometric and abstract spin nets and spin foams assumes that the couplings do not depend on the position or direction of the edges or plaquettes in the lattice, otherwise, one would need to introduce more decorations for the spin nets and foams. However, one can show [89], for instance, that the  $\mathbb{Z}_2$  Ising gauge model in 4D with varying couplings is universal. In other words, one can encode all other  $\mathbb{Z}_p$  lattice models. Here, it would be interesting to develop the equivalent spin foam picture and to see how a spin foam with additional decorations could encode other spin foam models.

### 3. Lattice Gauge Theories

The spin foam representation originated as a representation of partition functions for gauge theories [1–6, 11]. Such gauge theories can also be formulated with discrete groups [10, 52, 90], and for  $\mathbb{Z}_2$ , they give the Ising gauge models [10]. A spin foam representation for the Yang-Mills theories with continuous Lie groups has been presented in [91], see also [92, 93], and the results can be easily adapted to discrete groups. For the Abelian (discrete and continuous) groups, these are again closely related to the construction of dual models [78] and the high-temperature (or strong coupling) expansion [84].

Given a lattice, one associates group variables to its (oriented) edges. To be more precise, a lattice in this context is an orientable 2-complex  $\kappa$ , within which one can uniquely specify the 2-cells, called faces or plaquettes. The aim is to construct models with gauge symmetries at the vertices, where the gauge action is given by  $g_e \rightarrow g_{s(e)} g_e g_{t(e)}^{-1}$  and the  $g_v$  are the gauge symmetry parameters. They are associated to the vertices of the lattice; remember that  $s(e)$  is the source vertex of  $e$  and  $t(e)$  is the target vertex. Gauge-invariant quantities are given by the Wilson loops.<sup>11</sup> Expressing the action as a function of these quantities ensures that it in turn is gauge invariant, as can be easily checked. A Wilson loop is the trace (in some matrix representation) of the oriented product of group variables associated to a loop of edges in the lattice.<sup>12</sup> The “smallest” Wilson loops are those constructed from edges around a single lattice face. The associated face variables are then  $h_f =: \vec{\prod} g_e$ , where  $\vec{\prod}$  denotes the oriented product.

Given a unitary (finite-dimensional) matrix representation of a group  $g \mapsto U(g)$ , a Wilson-styled action is given by

$$S = \alpha \sum_f \Re \left( \text{tr} \left( U(h_f) \right) \right). \quad (9)$$

Here,  $\alpha$  is a coupling constant and  $U$  is a matrix representation of  $\mathbb{Z}_q$ . As we saw earlier, for the groups  $\mathbb{Z}_q$ , the irreducible unitary representations are 1 dimensional and labelled by  $k \in \{0, \dots, q-1\}$  so that the representations matrices are realized by  $g \mapsto \exp((2\pi i/q)k \cdot g)$ .

As before, let us generalize to a class of partition functions of the form

$$Z = \frac{1}{q^{\#e}} \sum_{g_e} \prod_f w_f(h_f), \quad (10)$$

where the weights  $w_f$  are class functions; that is, just like the trace, they are invariant under conjugation  $w_f(ghg^{-1}) = w_f(h)$ . Moreover, we included a measure normalization factor  $1/q$  for every summation over the group  $\mathbb{Z}_q$ .

**3.1. Spin Foam Representation.** The face weights  $w_f$ , as class functions, can be expanded in characters. For the Abelian groups  $\mathbb{Z}_q$ , this again just amounts to the discrete Fourier transform. Thus, the expansion takes the initial form

$$Z = \frac{1}{q^{\#e}} \sum_{g_e} \prod_f \sum_{k_f} \tilde{w}_f(k_f) \chi_{k_f}(h_f), \quad (11)$$

while the derivation of the spin foam representation proceeds as in Section 2 as follows:

$$\begin{aligned} Z &= \frac{1}{q^{\#e}} \sum_{g_e} \sum_{k_f} \left( \prod_f \tilde{w}_f(k_f) \chi_{k_f}(h_f) \right) \\ &= \frac{1}{q^{\#e}} \sum_{k_f} \left( \prod_f \tilde{w}_f(k_f) \right) \prod_e \left( \sum_{g_e} \prod_{f \supset e} \chi_{k_f}(g_e^{o(f,e)}) \right) \\ &= \sum_{k_f} \left( \prod_f \tilde{w}_f(k_f) \right) \prod_e \delta^{(q)} \left( \sum_{f \supset e} o(f,e) k_f \right) \\ &= \sum_{k_f} \left( \prod_f \tilde{w}_f(k_f) \right) \prod_v \prod_{e \supset v} \delta^{(q)} \left( \sum_{f \supset e} o(f,e) k_f \right), \end{aligned} \quad (12)$$

where  $o(f,e) = 1$  if the orientation of the edge coincides with the one induced by the “adjacent” face and  $-1$  otherwise. In the last step, we simply replace the Kronecker- $\delta$  weighting each edge by its square and associate one factor to each vertex. The reason is that in the spin foam representation one usually works with vertex amplitudes, which in this case is  $A_v = \prod_{e \supset v} \delta^{(q)} \left( \sum_{f \supset e} o(f,e) k_f \right)$ . Having said that, these amplitudes and the splitting are more complicated for both the non-Abelian groups (see Section 5) and continuous groups.

**3.2. Dual Models.** The construction of the dual models [78] proceeds by solving for the Gauss constraints associated to the edges in (12), that is, by solving the Kronecker- $\delta$ s. These enforce that the oriented sum of representation labels, associated to the faces incident at a given edge, vanishes.

(2d) For the lowest-dimensional case, namely, two dimensions, one obtains a “trivial” dual theory. Since every edge lies on just two faces, all of the representation labels coincide:  $k_f \equiv k$ , and one obtains (assuming that the  $\tilde{w}_f(k_f)$  do not depend on the position and orientation of the plaquettes,  $\tilde{w}_f(k_f) = \tilde{w}(k)$ )

$$Z = \sum_k (\tilde{w}_k)^{\#f}. \quad (13)$$

(3d) For the three-dimensional case, we have already seen that lattice models without any gauge symmetry are dual to lattice gauge models. For example, the Ising gauge model in 3d is dual to the 3d Ising model.

(4d) In four dimensions, lattice gauge models are dual to lattice gauge models. In the case that the cohomology of the lattice is nontrivial, topological models might appear as the dual model [94].

**3.3. High-Temperature Expansion and Spin Foams.** As for the Ising models, there is a high-temperature expansion, which for the Ising gauge model is also an expansion in powers of  $\tilde{w}(1)/\tilde{w}(0) = \tanh \beta$ . The discussion is very similar to the one for the Ising models, with the difference being that the perturbative contributions are now represented by closed surfaces rather than closed polymers.

The “ground state” is the configuration with  $k_f = 0$  for all faces. The Gauss constraints demand that the number of excited plaquettes (those with  $k_f \neq 0$ ) incident at every edge must be even. In particular, this means that the excited plaquettes must form closed surfaces. In the case of the Ising gauge model on a hypercubical lattice, the lowest-order contribution arises at  $(\tilde{w}(1)/\tilde{w}(0))^6$  from those configurations with excited faces on the boundary of a single 3d cube. As one might expect, there is a combinatorial factor arising from the number of embeddings of this cube into the ambient lattice.

More generally, the perturbative contributions are now described by closed, possibly branched surfaces. (For the free energy, one has only to consider connected components [95].) Here, a proper branching is a sequence of edges where at least three excited plaquettes meet. Excited plaquettes make up the elementary surfaces or spin foam faces. In other words, at the inner edges of these spin foam faces, there are always only two excited plaquettes meeting. Again the Gauss constraints demand that the representation labels of all of the plaquettes in a given spin foam face agree. For homogeneous and isotropic couplings, the spin foam amplitude (ignoring the combinatorial embedding factors) depends on the number of plaquettes  $a$  inside each spin foam face, but not on how the spin foam is embedded into the lattice. The contribution of a spin foam face with label  $k$  is  $\sim (\tilde{w}_k/\tilde{w}_0)^a$ . (In general, for models with the non-Abelian groups, the amplitudes might also depend on the

length of the branching or spin foam edges.) This definition of geometric and abstract spin foams [83] parallels the one given for spin nets. Note that the condition for abstract spin foams invalidates the conditions of the high-temperature expansion, which is an expansion in the number of excited plaquettes of the underlying lattice.

In Section 4, we will discuss the *BF*-theory. This is a topological model that satisfies the conditions necessary to class it as an abstract spin foam. In other words, the amplitude does not depend on the areas  $a$  of the spin foam faces. The reason is that  $\tilde{w}(k)/\tilde{w}(0) \equiv 1$  for the *BF*-models. For the Ising gauge model, this is the case at zero temperature.<sup>13</sup>

**3.4. Expansion around a Topological Sector.** If one starts from the first line in (11) and keeps the summation over both group and representation labels, one obtains

$$\begin{aligned} Z &= \frac{1}{q^{\sharp e}} \sum_{g_e} \sum_{k_f} \left( \prod_f \tilde{w}_f(k_f) \chi_k(h_f) \right) \\ &= \frac{1}{q^{\sharp e}} \sum_{g_e} \sum_{k_f} \left( \prod_f \tilde{w}_f(k_f) \exp\left(\frac{2\pi i}{q} k_f \cdot h_f(g_e)\right) \right) \\ &= \frac{1}{q^{\sharp e}} \sum_{g_e} \sum_{k_f} \left( \prod_f \exp\left(\frac{2\pi i}{q} k_f \cdot h_f(g_e) + \ln \tilde{w}_f(k_f)\right) \right). \end{aligned} \quad (14)$$

This corresponds to a first-order representation of the Yang-Mills theory where the  $g_e$  are the connection variables and the  $k_f$  represent the dual (electric field or flux) variables. The model where all  $\tilde{w}_f(k_f) = 1$  is a topological *BF*-model, whose partition function can be solved exactly. Hence, the Yang-Mills theory can be understood as a deformed *BF*-theory [92, 93, 96, 97]. The deformation appears here as the  $\ln(\tilde{w}_f(k_f))$  term; in the continuum, it is  $B \wedge \star B$ .<sup>14</sup>

The expansion of models with propagating degrees of freedom, for instance, those modelling  $4d$  gravity and the Yang-Mills theories, around topological theories has been discussed, for instance, in [83, 98, 99].

In the case of the Ising model, one expands around the *BF*-theory partition function by introducing an expansion parameter  $\alpha = \tanh \beta - 1$ , which is small for low temperatures:

$$\begin{aligned} Z &= (e^{\beta/2} \cosh \beta)^{\sharp f} \sum_{k_f} \prod_e \delta^{(2)}\left(\sum_{f \supset e} o(f, e) k_f\right) \\ &\quad \times \left( \prod_f (\delta^{(2)}(k_f) + (1 + \alpha) \delta^{(2)}(k_f - 1)) \right) \\ &= (e^{\beta/2} \cosh \beta)^{\sharp f} \sum_{k_f} \prod_e \delta^{(2)}\left(\sum_{f \supset e} o(f, e) k_f\right) \end{aligned}$$

$$\begin{aligned} &\times \left( 1 + \alpha \sum_{f_1} \delta^{(2)}(k_{f_1} - 1) + \alpha^2 \right. \\ &\quad \times \sum_{f_1 < f_2} \delta^{(2)}(k_{f_1} - 1) \delta^{(2)}(k_{f_2} - 1) + \dots + \alpha^{\sharp f} \\ &\quad \times \sum_{f_1 < \dots < f_{\sharp f}} \delta^{(2)}(k_{f_1} - 1) \dots \delta^{(2)}(k_{f_{\sharp f}} - 1) \left. \right). \end{aligned} \quad (15)$$

Note that to get to the second equality we have used the fact that  $\delta^{(2)}(k_f) + (1 + \alpha) \delta^{(2)}(k_f - 1) = 1 + \alpha \delta^{(2)}(k_f - 1)$ . Thus, for a given configuration  $\{k_f\}_f$ , the coefficient of  $\alpha^1$  records the number of excited faces. The coefficient of  $\alpha^2$  gives the number of ordered pairs of excited faces. The next coefficient records the number of ordered triples of faces and so on. The last coefficient of  $\alpha^{\sharp f}$  is one for the configuration  $k_f \equiv 1$  and zero for all other configurations. The terms in this expansion can be seen as expectation values of observables in a *BF*-model, where the observables are the numbers of ordered  $n$ -tuples of excited plaquettes, for each  $0 \leq n \leq \sharp f$ .

## 4. Topological Models

In the previous section, we saw that theories of the Yang-Mills type can be understood as a deformation of the *BF*-theories, which we will discuss here in more detail. From the conventional standpoint,  $F$  represents the curvature of a connection, usually taking values in the Lie group, while  $B$  is a Lie algebra-valued  $(D-2)$ -form. Hence, in constructing these models for discrete groups, the following question arises: with what should one replace the  $B$  variables? We have seen that the  $B$ -field corresponds to the representation labels. Examples for topological models with discrete groups are discussed in several texts [51, 53, 100, 101]. See also [102, 103] for the relation between the Chern-Simons-like theories and the *BF*-like theories in  $3d$  with discrete group  $\mathbb{Z}_q$ . In this section, we will restrict to the Abelian groups, in particular  $\mathbb{Z}_q$ . The models for the non-Abelian groups will be presented in Section 5.

The *BF*-theory is a topological field theory in which the equations of motion demand that the “local” curvature vanishes. One often starts with the partition function encoding this requirement. It is defined in a manner very similar to that of the previous sections (see, for instance, [6, 104]) as follows:

$$Z = \frac{1}{q^{\sharp e}} \sum_{g_e} \prod_f \delta^{(q)}(h_f(g_e)), \quad (16)$$

where again  $h_f(g_e) = \vec{\Pi}_{e \subset f} g_e$  means the oriented product<sup>15</sup> of edges around a face. Note that this is just a special case of the gauge theory partition functions (10), obtained by setting  $w_f = \delta^{(q)}$ .

The partition function can be easily evaluated:

$$Z = \# \left[ \text{configurations } \{g_e\}_e : \vec{\Pi}_{e \subset f} g_e = \text{id } \forall f \right] \times \frac{1}{q^{\sharp e}}. \quad (17)$$

For the groups  $\mathbb{Z}_q$ , an expansion of the type detailed in (14) yields

$$\begin{aligned} Z &= \frac{1}{q^{\#e}} \sum_{g_e} \prod_f \delta^{(q)}(h_f(g_e)) \\ &= \frac{1}{q^{\#e + \#f}} \sum_{g_e} \sum_{k_f} \exp\left(\frac{2\pi i}{q} \sum_f k_f \cdot h_f(g_e)\right). \end{aligned} \quad (18)$$

One may interpret the term in the exponential as a  $BF$ -theory action, where the representation labels  $k_f$  represent the  $B$ -field on the lattice, while the products  $h_f(g_e) = \sum_{e \subset f} o(f, e) g_e$  represent curvature.

The spin foam representation can be derived in the same way as for the gauge theories in Section 3.1:

$$Z = \frac{1}{q^{\#f}} \sum_{k_f} \prod_v \left( \prod_{e \supset v} \delta^{(q)} \left( \sum_{f \supset e} o(f, e) k_f \right) \right). \quad (19)$$

This recasts the partition function as

$$Z = \# \left[ \text{configurations } \{k_f\}_f : \sum_{f \supset e} o(f, e) k_f \equiv 0 \pmod{q} \quad \forall e \right] \times \frac{1}{q^{\#f}}, \quad (20)$$

that is, as the number of assignments  $\{k_f\}_f$  satisfying the Gauss constraint for every edge. In the following section, we will discuss the local symmetries of the  $BF$ -theory partition function. We will find that, on the set of all configurations  $\{k_f\}_f$ , there acts a translation symmetry. This is realized at the 3-cells of the lattice. Hence,  $Z$  reflects the orbit volume of this translational gauge symmetry. It is this symmetry that leads to divergences in the  $BF$ -theory partition functions for the “continuous” Lie groups [105, 106].<sup>16</sup>

**4.1. Symmetries of the Partition Function.** We will now discuss the gauge symmetries of the partition function (16). As its form coincides with that of the gauge theories (10), the partition function is invariant under the usual gauge transformations  $g_e \rightarrow g_{s(e)} g_e g_{t(e)}^{-1}$ , where  $g_v \in \mathbb{Z}_q$  are gauge parameters associated to the vertices. This is manifest in the representation given in (16).

There is a further symmetry, the so-called translation symmetry, which is easiest to see in the spin foam representation. This symmetry is based on the 3-cell  $c$  of the lattice, that is, the cubes for a hypercubical lattice.<sup>17</sup> Consider a field  $c \mapsto k_c$  of gauge parameters associated to the cubes and define the gauge transformations

$$k'_f \equiv k_f + \sum_{c \supset f} o(c, f) k_c \pmod{q}, \quad (21)$$

where  $o(c, f) = 1$  if the orientation of  $f$  agrees with that induced by  $c$  and  $-1$  otherwise. If  $\{k_f\}_f$  is a configuration

that satisfies all of the Gauss constraints, then so does  $\{k'_f\}_f$  and vice versa. The reason stems from the following fact: if an edge  $e$  is in the boundary of a 3-cell  $c$ , then there are exactly two faces, say  $f_1$  and  $f_2$ , that are both in the boundary of  $c$  and adjacent to  $e$ . The orientation factors are such that the contributions of  $k_c$  to the two faces  $f_1$  and  $f_2$  are of opposite signs in the Gauss constraint associated to  $e$ . Hence, the stated result follows and the contributions of these two configurations,  $\{k_f\}_f$  and  $\{k'_f\}_f$ , to the partition function are equal.

The above symmetry is a realization of the well-known cohomological principle that “the boundary of a boundary is zero” ( $\partial \circ \partial = 0$ ) and underlies the Bianchi identity for the curvature. With the Bianchi identity, one can also explain the translation symmetry; see [39, 49].

In  $3d$ , the  $BF$ -theory with  $SU(2)$  as its gauge group has a gravitational interpretation. This translation symmetry corresponds to the diffeomorphism symmetry of the theory. The gauge field  $k_c$  corresponds in the dual lattice to a field associated to the dual vertices, and one can interpret the gauge transformation as a translation<sup>18</sup> of these dual vertices (which in the gravity models are the vertices in a triangulation of space time).

In  $4d$ , the 3-cells are dual to edges in the dual lattice, so this symmetry translates the dual edges rather than the dual vertices. For gravity-like models, on the other hand, one would like to break down this translation symmetry, where it is based on the dual edges, to symmetries based on the dual vertices. (Correspondingly, in  $4d$ , diffeomorphism symmetry of the action leads to the Noether charges encoding the contracted Bianchi identities and not the Bianchi identity itself.) In the case of gravity, one strategy is to impose the so-called simplicity constraints [56–61, 64] within the partition function. For models with finite groups, the question arises as to whether there is a class of  $4d$  gauge models with “translation-like” symmetries based on the dual vertices. Such models would be nontopological and feature propagating degrees of freedom, as a symmetry group based on dual vertices is much smaller than that for  $BF$ , where it is based on dual edges. See also the discussion in Sections 5 and 6.

## 5. Gauge Theories for the Non-Abelian Groups

In the following, we will consider how to generalize the construction performed so far for the finite, but non-Abelian, groups  $G$ . Details about representations can be found in [79].

Again, we denote by  $\rho$  the irreducible, unitary representations of  $G$  on  $\mathbb{C}^n \simeq V_\rho$ , where  $n = \dim \rho$ . Note that for the non-Abelian groups  $n$  can be larger than 1. Every function  $f : G \rightarrow \mathbb{C}$  can be decomposed into matrix elements of representations; that is,

$$f(g) = \sum_\rho \sqrt{\dim \rho} \tilde{f}_{\rho ab} \rho(g)_{ab}, \quad (22)$$



where the sum ranges over all “equivalence classes” of irreducible representations of  $G$ . The  $\tilde{f}_\rho$  are given by

$$\tilde{f}_{\rho,a,b} = \frac{1}{|G|} \sum_g \sqrt{\dim \rho} f(g) \rho(g)_{ab}, \quad (23)$$

where  $|G|$  denotes the number of elements in the group  $G$ . Introducing an inner product between functions  $f_1, f_2 : G \rightarrow \mathbb{C}$  by

$$\langle f_1 | f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g) = \sum_{\rho,a,b} \overline{(\tilde{f}_1)_{\rho,a,b}} (\tilde{f}_2)_{\rho,a,b}, \quad (24)$$

then the matrix elements of  $\rho$  are orthogonal; that is,

$$\langle \rho_{ij} | \tilde{\rho}_{kl} \rangle = \frac{\delta_{\rho,\tilde{\rho}} \delta_{ik} \delta_{jl}}{\dim \rho}. \quad (25)$$

Furthermore, we denote by

$$\chi_\rho(g) = \text{tr}(\rho(g)) \quad (26)$$

the character of  $\rho$ ; then the  $\delta$ -function on the group  $G$  is given by

$$\delta_G(g) = \sum_\rho \dim \rho \chi_\rho(g), \quad (27)$$

which satisfies

$$\frac{1}{|G|} \sum_g \delta_G(g) f(g) = f(1). \quad (28)$$

**5.1. The  $G$ -Gauge Theory on a Two-Complex  $\kappa$ .** Consider an oriented two-complex<sup>19</sup>  $\kappa$ . Denote the set of edges  $E$  and the set of faces  $F$ ; then by a *connection* we mean an assignment  $E \rightarrow G$  of group elements  $g_e$  to edges  $e \in E$ . For every face  $f \in F$ , we denote the *curvature* of this connection by

$$h_f := \prod_{e \in \partial f}^{\rightarrow} g_e^{o(f,e)} \quad (29)$$

which is the ordered product of group elements of edges in the boundary of  $f$ , where we take  $o(f,e) = \pm 1$  depending on the relative orientation of  $e$  and  $f$ . In the non-Abelian case, the ordering of the group elements around a face/plaquette actually matters, and we choose here the convention as depicted in Figure 1.

As before, we define a gauge-invariant partition function by choosing a collection of weight-functions  $w_f : G \rightarrow \mathbb{C}$  invariant under conjugation. These encode the action “and the path integral measure” of the system. The partition function is defined as

$$Z = \frac{1}{|G|^{\sharp E}} \sum_{g_e} \prod_{f \in F} w_f(h_f), \quad (30)$$

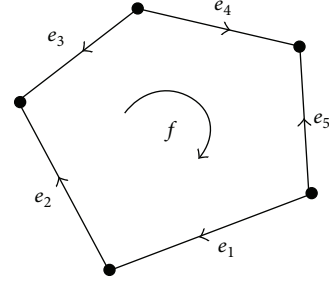


FIGURE 1: A face  $f$ , bordered by edges  $e_1, \dots, e_5$ . The curvature is given by  $h_f = g_{e_5}^{-1} g_{e_4} g_{e_3}^{-1} g_{e_2} g_{e_1}$  (note the ordering).

where  $|G|$  is the number of elements in  $G$ . Since  $w_f$  can be expanded into characters via (22), the path integral can be written as

$$Z = \frac{1}{|G|^{\sharp E}} \sum_{g_e} \sum_{\rho_f} \prod_f (\tilde{w}_f)_{\rho_f} \rho_f(g_{e_1}^{\pm 1})_{i_1 i_2} \times \rho_f(g_{e_2}^{\pm 1})_{i_2 i_3} \cdots \rho_f(g_{e_n}^{\pm 1})_{i_n i_1}. \quad (31)$$

In sum, there is for each edge  $e$  a representation  $\rho_f(g_e)$  appearing for every face  $f$  adjacent to  $e$ ,  $f \supset e$ . The corresponding sum results in (assuming that the orientations of all  $f$  agree with those of  $e$  for the moment, in order not to overburden the notation)

$$(P_e)_{i_1 i_2 \dots i_n; j_1 j_2 \dots j_n} := \frac{1}{|G|} \sum_{g \in G} \rho_{f_1}(g)_{i_1 j_1} \rho_{f_2}(g)_{i_2 j_2} \cdots \rho_{f_n}(g)_{i_n j_n}. \quad (32)$$

It is not hard to see that the operator  $P$ , called the *Haar-intertwiner*, given by

$$(P_e \psi)_{i_1 i_2 \dots i_n} := (P_e)_{i_1 i_2 \dots i_n; j_1 j_2 \dots j_n} \psi_{j_1 j_2 \dots j_n}, \quad (33)$$

which maps the edge-Hilbert space

$$\mathcal{H}_e := V_{\rho_1} \otimes V_{\rho_2} \otimes \cdots \otimes V_{\rho_n} \quad (34)$$

to itself, is actually an orthogonal projector on the gauge-invariant subspace of  $\mathcal{H}_e$ .<sup>20</sup> Note that, while in the Abelian case one has to sum over all representations over faces such that at all edges the “oriented” sum adds up to zero (as in (12)), in the non-Abelian generalization one has to sum over all representations such that at each edge  $e$  the representations of the faces  $f \supset e$  meeting at  $e$  have to contain, in their tensor product, the trivial representation. Also, one obtains a nontrivial tensor  $P_e$  for each edge, which in the case of the Abelian gauge groups is just the  $\mathbb{C}$ -number  $\delta_{\pm k_1 \pm k_2 \pm \dots \pm k_n, 0}$ . Since the representations for the non-Abelian groups can be more than 1 dimensional, in general the tensor  $P_e$  has indices which are contracted at each vertex  $v$  in the two-complex  $\kappa$ .

Choosing an orthonormal base  $t_e^{(k)}$ ,  $k = 1, \dots, m$  for the invariant subspace of  $\mathcal{H}_e$ , we get

$$P_e = \sum_{k=1}^m |t_e^{(k)}\rangle \langle t_e^{(k)}|. \quad (35)$$

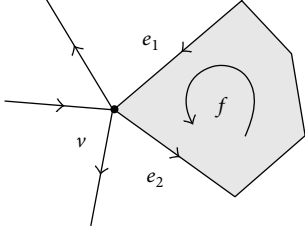


FIGURE 2: The relative orientations of both  $e_1$  to  $f$  and  $e_2$  to  $f$  agree, so in both Hilbert spaces  $H_{e_1}$  and  $H_{e_2}$ , the factor  $V_{\rho_f}$  occurs. Moreover  $|i_{e_1}\rangle$  and  $\langle i_{e_2}|$  have indices belonging to  $V_{\rho_f}$  in opposite positions, so they may be contracted.

In case the orientations of  $e$  and  $f$  do not agree for some  $e$ , then  $g_e$  is essentially replaced by  $g_e^{-1}$  in (32), which leads to the appearance of the dual<sup>21</sup> representation in the tensor product (34) of the  $\mathcal{H}_e$ . In this case,  $\iota_e^{(k)}$  labels a basis of intertwiner maps between the tensor product of all representations associated to faces  $f$  with  $o(f, e) = -1$  and the tensor product of all representations associated to the faces with  $o(f, e) = 1$ .

In (35), one regards  $|i_e\rangle$  and  $\langle i_e|$  as attached to, respectively, the endpoint and the beginning of  $e$ . For each vertex  $v$  in  $\kappa$ , the associated  $|i_e\rangle$  and  $\langle i_e|$  can be contracted in a canonical way. For every face  $f$  which touches  $v$ , there are exactly two edges  $e_1, e_2$  in the boundary of  $f$  that meet at  $v$ . The definitions above are exactly such that, if one chooses a basis in each  $V_\rho$  and the dual basis in each  $V_\rho^*$ , in  $\iota_{e_1}$  and  $\iota_{e_2}$  the two indices associated to  $f$  are in opposite position, so they can be contracted. See Figure 2 for an example.

Therefore, contracting all the appropriate  $\iota_e$  at one vertex leaves one with the vertex-amplitude  $\mathcal{A}_v(\rho_f, \iota_e)$ , which depends on the representations  $\rho_f$  and intertwiners  $\iota_e$  associated to the faces and edges that meet at  $v$  (and the orientations of these). The vertex amplitude can be computed by evaluating the so-called *neighbouring spin network function*, living on graph which results from a dimensional reduction of the neighbourhood of  $v$ . Construct a spin network, where there is a vertex for each edge  $e$  touching  $v$ , and a line between any two vertices for each face  $f$  between two edges. Assigning the  $\rho_f$  to the lines of the spin network and the intertwiners  $|i_e\rangle$  or  $\langle i_e|$  to the vertices, depending on whether the corresponding edge  $e$  is incoming or outgoing of  $v$ , results in a spin network, whose evaluation gives  $\mathcal{A}_v$ .

With this, the state sum can, using (31) and (32), be written in terms of vertex amplitudes via

$$Z = \sum_{\rho_f \iota_e} \prod_f (\tilde{w}_f)_\rho \prod_v \mathcal{A}_v(\rho_f, \iota_e). \quad (36)$$

**5.2. Examples.** The first example we consider is the  $G$  BF-theory, which corresponds to an integral over only flat

connections, that is, the choice  $w_f(h) = \delta_G(h)$ . Since the  $\delta$ -function can be decomposed into characters with  $(\tilde{w}_f)_\rho = \dim \rho_f$ , we get

$$\mathcal{Z} = \sum_{\rho_f \iota_e} \prod_f \dim \rho_f \prod_v \mathcal{A}_v. \quad (37)$$

If  $\kappa$  is the two-complex dual to a triangulation of a 3-dimensional manifold, then the vertex amplitude is essentially the analogue of the  $6j$ -symbol for  $G$ <sup>22</sup>.

The second example that one usually considers is the Yang-Mills theory. The Wilson action can be specified, similar to the Abelian case, by choosing a unitary representation  $\bar{\rho}$ , so that

$$S_{YM}(h) = \frac{\alpha}{2} (\chi_{\bar{\rho}}(h) + \chi_{\bar{\rho}}(h^{-1})) = \alpha \operatorname{Re}(\chi_{\bar{\rho}}(h)), \quad (38)$$

where  $\alpha$  is the coupling constant. The weights are then given by  $w_f(h) = \exp(-S_{YM}(h))$ .

**5.3. Constrained Models.** There is a generalization of the state-sum models (36), coming from the desire to obtain nontopological generalizations of the BF-theory. It amounts to choosing the intertwiners  $\iota_e$  not to span all of the invariant subspace, but only a proper subspace  $V_e \subset \operatorname{Inv}(\mathcal{H}_e)$ . This originates in the attempt to define a state-sum model for general relativity, which can be written as a constrained BF-theory. The subspace  $V_e$  is viewed as the space of intertwiners satisfying the so-called simplicity constraints; see, for example, [56, 64, 107]. Examples for such models are the Barrett-Crane model [57] and the EPRL and FK models [58–61]. These models can also be written in the form of a path integral over connections [91, 108, 109], but we will not concern ourselves with this here.

In the following, we will introduce a class of models which can be seen as the generalization of the Barrett-Crane models to finite groups. Given a finite group  $H$ , the model is a state-sum model for the group  $G = H \times H$ . Irreducible representations of  $G$  are then pairs of irreducible representations  $(\rho_f^+, \rho_f^-)$  of  $H$ . In the edge-Hilbert space  $\mathcal{H}_e$ , there is a specific element  $\iota_{BC}$  of the space of gauge-invariant elements, called the *Barrett-Crane intertwiner*. It is only nonzero if and only if the representations  $\rho_f^+$  and  $\rho_f^-$  are dual to each other. In that case, for an edge  $e$  with attached faces  $f_k$ , consider the Hilbert space

$$\tilde{\mathcal{H}}_e = V_{\rho_{f_1}^+} \otimes \cdots \otimes V_{\rho_{f_k}^+}. \quad (39)$$

Again, we have assumed that the orientations of all faces  $f_k$  and  $e$  agree, in order not to overburden the notation; otherwise, replace  $V_{\rho_f^+}$  by its dual or, equivalently, exchange  $\rho_f^\pm$ . If we consider the projector  $\tilde{P}_e : \tilde{\mathcal{H}}_e \rightarrow \tilde{\mathcal{H}}_e$  onto the subspace of elements invariant under the action of  $H$ , then  $\tilde{P}_e$  can be seen as an element of  $\tilde{\mathcal{H}}_e \otimes \tilde{\mathcal{H}}_e^*$ , which is naturally isomorphic to  $\mathcal{H}_e$ , because  $\rho_f^\pm$  are dual to each other. The corresponding element  $\iota_{BC}$  in  $\tilde{\mathcal{H}}_e$  is invariant under  $H \times H$ , as one can readily see, and it is our distinguished element onto which  $P_e$  is projecting.

Since the projection space for each edge is (at most) 1 dimensional, the vertex amplitude for the BC-model does not depend on intertwiners, but only on the representations  $\rho_f^+ = (\rho_f)^*$  associated to the faces. In fact, since the representations are unitary, every representation is dual to itself, so we effectively have only one representation  $\rho_f$  attached to each face. Using diagrammatical calculus [110], it is not hard to show that the vertex amplitude for the BC model is given by a sum over squares of the *BF*-theory model on the same vertex: that is,

$$\mathcal{A}_v^{(BC)}(\rho_f^\pm, t_{BC}) = \sum_{t_e} \left| \mathcal{A}_v^{(BF)}(\rho_f, t_e) \right|^2, \quad (40)$$

where the sum ranges over an orthonormal basis  $t_e$  of  $S_3$ -intertwiners in  $\mathcal{H}_e$  for every edge  $e$  at the vertex  $v$ .

Let us shortly discuss the case of  $H = S_3$ , the group of permutations in three elements. For  $S_3$ , which is generated by (12), the permutation of the first two elements, and (123), which is the cyclic permutation, subject to the relation  $(12)^2 = (123)^3 = 1$ , the representations theory is well known [79]: There are three irreducible representations, called [1],  $[-1]$ , and [2]. The first is the trivial representation and the second one maps a permutation  $\sigma$  to its sign  $(-1)^\sigma$ . The third one is two dimensional, and

$$\rho((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho((123)) = \begin{pmatrix} \frac{\cos 2\pi}{3} & -\frac{\sin 2\pi}{3} \\ \frac{\sin 2\pi}{3} & \frac{\cos 2\pi}{3} \end{pmatrix}. \quad (41)$$

The nontrivial tensor products of the representations decompose as

$$\begin{aligned} [-1] \otimes [-1] &= [1], & [-1] \otimes [2] &= [2], \\ [2] \otimes [2] &= [1] \oplus [-1] \oplus [2]. \end{aligned} \quad (42)$$

Therefore, the intertwiner space in  $\mathcal{H}_e$  is, for example, three dimensional, when there are four faces attached to  $e$  carrying the representation [2]. Hence, for  $H = S_3$ , the BC-model is an example for a constrained version of an  $H \times H$ -state-sum model, as defined in the last section, because  $P_e$  projects onto a proper subspace of the invariant subspace of  $\mathcal{H}_e$ .

This class of models is an example for an abstract spin foam [83], since its amplitudes only depend on combinatorial data, and the topology of the two-complex  $\kappa$ . In particular, it leads to a model which is invariant under refinement of the two-complex  $\kappa$  by trivial subdivisions of edges or faces. Hence, these models provide potential examples for models which are background independent, but not topological (as is the case in the full theory).

Note that there is a wealth of different models, which come from different choices of nontrivial subspaces of  $\mathcal{H}_e$ , onto which  $P_e$  is projecting. In particular, one could consider EPRL-like models where  $G = S_4 \times S_4$  or  $G = A_4 \times A_4$  and consider the subspace of intertwiners for  $\rho_f^+ = \rho_f^-$ , which are in the image of a boosting map  $b : V_{\rho_f}^+ \rightarrow V_{\rho_f}^+ \otimes V_{\rho_f}^-$ ,

given by the fusion coefficients (see, e.g., [58–61] for  $H = SU(2)$ ). Here,  $S_4$  ( $A_4$ ) is the group of “even” permutation of four elements. Another, possibly more geometric way would be to consider embeddings  $S_4 \rightarrow S_5$  and thus construct proper subspaces of  $S_5$ -intertwiners.<sup>23</sup> Such models can be considered as truncations of the EPRL models as  $S_4$  ( $A_4$ ) is the “chiral” symmetry group of the tetrahedron and a subgroup of the rotation group. Here, it will be interesting to investigate the relation to the full models in more detail.

The models can serve to test many proposals for the full theory, for instance, how the different choices of edge projectors  $P_e$  determine the physical degrees of freedom or particle content of the given theory. This is related to the problem of implementing the simplicity constraints into spin foam models. We are planning to investigate the features of these models, in particular their symmetries [65a, 65b] and behaviours under coarse graining, in future work.

## 6. The Hilbert Space, the Transfer Operators, and Constraints

We intend to derive the *transfer operators* [111, 112] for the Yang-Mills-like gauge theories, having partition functions of the forms (10) and (30) and incorporating a generic finite group  $G$  of cardinality  $|G|$ . We will permit  $G$  to be non-Abelian. The first part of the discussion is of a similar nature to that found in [112] for the Yang-Mills theory. However, we will also include the *BF*-theory as a special case. From the transfer operators, one can obtain, via a limiting procedure, the Hamiltonian operators. However, for the *BF*-theory, we will see that this limiting procedure is not necessary. Rather, the transfer operators can be understood as projectors onto the space of the so-called physical states. These can be characterized as lying in the kernel of the “Hamiltonian” constraints. This illustrates the general principle that a path integral with local symmetries acts a projector onto a constrained subspace [113, 114]. Conversely, one can construct a projector onto the constrained subspace as a path integral. See [115], in which this is performed for  $3d$  gravity in its formulation as an  $SU(2)$  *BF*-theory.

The transfer operator  $\hat{T}$  is defined on a Hilbert space  $\mathcal{H}$ , so that the partition function can be written as

$$Z = \text{tr}_{\mathcal{H}} \hat{T}^N. \quad (43)$$

We assume for simplicity<sup>24</sup> that the lattice is hypercubical. One lattice direction is designated as a time direction, in which there are periodic boundary conditions. The trace  $\text{tr}_{\mathcal{H}}$  is the trace over states in the following Hilbert space. It is associated to the spatial lattice and is built as a direct product of the Hilbert spaces associated to the spatial edges  $e_s$ . More succinctly written as  $\mathcal{H} = \bigotimes_{e_s} \mathcal{H}_{e_s}$ , the “configuration” variable attached to any given edge is a group element, so the associated Hilbert space is the space of complex functions on the group, equipped with an inner product,<sup>25</sup> that is,  $\mathcal{H}_e = \mathbb{C}[G]$ .

We will work in the representation in which a basis of eigenstates is given by  $|g_e\rangle$ . This is known as the holonomy or

connection representation. For any unitary matrix representation,  $g_e \mapsto \rho(g_e)_{ab}$  of the group  $G$ , these are simultaneous eigenstates of the so-called edge holonomy operators  $\hat{\rho}(g_e)_{ab}$ :

$$\hat{\rho}(g_e)_{ab} |g\rangle = \rho(g_e)_{ab} |g\rangle. \quad (44)$$

These basis states are orthonormal:

$$\langle g' | g \rangle = \prod_{e_s} \delta^{(G)}(g'_s, g_s), \quad (45)$$

where  $\delta_G$  is the delta function on the group such that  $(1/|G|) \sum_g \delta^{(G)}(g, g') = 1$ . We have also the completeness relation

$$\text{Id}_{\mathcal{H}} = \frac{1}{|G|^{\#e_s}} \sum_{g_s} |g\rangle \langle g|. \quad (46)$$

We use this resolution of identity  $N$  times in (43) to rewrite the partition function as

$$Z = \text{tr}_{\mathcal{H}} \hat{T}^N = \frac{1}{|G|^{\#e_s}} \sum_{g_s, n} \prod_{n=0}^N \langle g_{n+1} | \hat{T} | g_n \rangle, \quad (47)$$

where we introduced  $n = 0, \dots, N$  to label the “constant time hypersurfaces” in the lattice. On the other hand, we will assume that our partition function is of the form

$$\begin{aligned} Z &= \frac{1}{|G|^{\#e}} \sum_{g_e} \prod_f w_f(h_f) \\ &= \frac{1}{|G|^{\#e}} \prod_{n=0}^N \prod_{(f_s)_n} w_{f_s}^{1/2}(h_{f_s}) \\ &\quad \times \prod_{(f_t)_n} w_{f_t}(h_{f_t}) \prod_{(f_s)_n} w_{f_s}^{1/2}(h_{f_s}). \end{aligned} \quad (48)$$

Here, we introduced the weight functions  $w_f$  of the holonomies around plaquettes  $h_f$ , which in the form of the right-hand side can also be chosen to differ for spatial plaquettes  $f_s$  and timelike plaquettes  $f_t$ . (This is necessary if one wants to obtain the Hamiltonian in the limit of small lattice constant in timelike direction.) Since we are dealing with a gauge theory, the weights  $w_f$  are class functions, that is, invariant under conjugation; furthermore, we will assume that  $w_f(h) = w_f(h^{-1})$ ; that is, the weights are independent of the orientation of the face. A weight function can therefore be expanded into characters, which are linear combinations of matrix elements of representations (in fact, characters are just the trace). Hence, the weight functions will be quantized as edge holonomy operators.

On the right-hand side of (48), we have just split the product into factors labelled by the time parameter  $n$ . Here, we assume that the plaquettes  $(f_t)_n$  are the ones between time  $n$  and  $n+1$ . Comparing the two forms of the partition functions (47) and (48), we can conclude that

$$\begin{aligned} \langle g_{n+1} | \hat{T} | g_n \rangle &= \prod_{f_s} w_{f_s}^{1/2}(h_{(f_s)_{n+1}}) \frac{1}{|G|^{\#e_t}} \\ &\quad \times \sum_{g_{e_t}} \prod_{f_t} w_{f_t}(h_{(f_t)_n}) \prod_{f_s} w_{f_s}^{1/2}(h_{(f_s)_n}). \end{aligned} \quad (49)$$

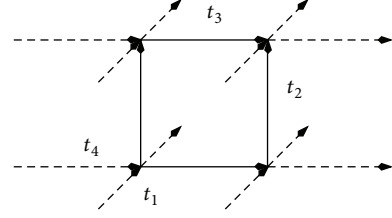


FIGURE 3: A timelike plaquette in a cubical lattice. The group elements at the timelike edges can be understood to act as gauge transformations on the vertices either at time steps  $n$  or  $n+1$ . Integration over the group elements associated to the timelike edges enforces therefore (via group averaging) a projector onto the gauge invariant Hilbert space.

The two outer factors can be easily quantized as multiplication operators, so that we can write

$$\hat{T} = \hat{W} \hat{K} \hat{W} \quad (50)$$

with

$$\hat{W} = \prod_{f_s} \hat{w}_{f_s}^{1/2}(h_{f_s}), \quad (51)$$

$$\langle g_{n+1} | \hat{K} | g_n \rangle = \frac{1}{|G|^{\#e_t}} \sum_{g_{e_t}} \prod_{(f_t)_n} w_{f_t}(h_{(f_t)_n}).$$

To tackle the operator  $\hat{K}$ , note that the group elements associated to the timelike edges appearing in the plaquette weights

$$w_{f_t}(g_{(e_s)_n} g_{(e_t)_n} g_{(e_s)_{n+1}}^{-1} g_{(e_t)_n}^{-1}) \quad (52)$$

can be understood as acting as gauge transformations associated to the vertices of the spatial lattice; see Figure 3.

That is, we define operators  $\hat{\Gamma}(\gamma)$ ,  $\gamma \in G^{\#v_s}$  by

$$\hat{\Gamma}(\gamma) |g\rangle = |\gamma^{-1} \triangleright g\rangle, \quad (53)$$

where  $(\gamma \triangleright g)_e = \gamma_{s(e)} g_e \gamma_{t(e)}^{-1}$  and  $s(e), t(e)$  are the source and target vertices of the edge  $e$ , respectively. The operators  $\hat{\Gamma}$  generate gauge transformations as

$$\langle g | \hat{\Gamma}(\gamma) | \psi \rangle = \langle \gamma \triangleright g | \psi \rangle = \psi(\gamma \triangleright g). \quad (54)$$

Now, we can see the plaquette weight in (52) as either a wave function at time  $n$  or a wave function at time  $n+1$  on which the group elements associated to the timelike edges act as gauge transformations. The sum in (51) over the group elements  $g_{e_t}$  then induces an averaging over all gauge transformations, that is, a projection onto the space of gauge-invariant states. Hence, we can write

$$\hat{K} = \hat{P}_G \hat{K}_0 = \hat{K}_0 \hat{P}_G, \quad (55)$$

where

$$\hat{P}_G = \frac{1}{|G|^{\#v_s}} \sum_{\gamma} \hat{\Gamma}(\gamma), \quad (56)$$

$$\langle g_{n+1} | \hat{K}_0 | g_n \rangle = \prod_{f_t} w_{f_t}(g_{(e_s)_n} g_{(e_s)_{n+1}}^{-1}).$$



The operator  $\hat{P}_G$  is a projector; that is,  $\hat{P}_G^2 = \hat{P}_G$ . One can easily show that  $\hat{\Gamma}(\gamma)\hat{P}_G = \hat{P}_G\hat{\Gamma}(\gamma) = \hat{P}_G$  for any  $\gamma \in G^{\sharp v_s}$ ; hence, it projects onto the space of gauge-invariant functions. As the operator  $\hat{W}$  is made up from gauge-invariant plaquette couplings, it commutes with  $\hat{P}_G$ , so that we can write

$$\hat{T} = \hat{P}_G \hat{W} \hat{K}_0 \hat{W} \hat{P}_G. \quad (57)$$

Here, we see an example for a general mechanism, namely, that the transfer operator for a partition function with local gauge symmetries acts as a projector onto the space of gauge-invariant states. We can characterize such gauge-invariant states as being annihilated by constraint operators. In the case of the usual lattice gauge symmetries, these constraints are the Gauss constraints, which we will discuss later on.

To discuss the action of  $K_0$ , we introduce first the spin network basis, in which  $\hat{K}_0$  will be diagonal.

**6.1. Spin Network Basis.** So far, we encountered the holonomy operators, that is, the multiplication operators  $\hat{\rho}(g_e)_{ab}$  in the configuration representation. The conjugated operators are the electric fields or fluxes, which act as “matrix” multiplication operators in the spin network basis. This is a convenient basis to discuss the remaining part  $K_0$  of the transfer operator.

To obtain the spin network basis [11, 12], we just need to use that every function on the group can be expanded as a sum over the matrix elements  $\rho(\cdot)_{ab}$  of all irreducible unitary representations  $\rho$ . For the Abelian groups, all irreducible representations are one dimensional; hence,  $a, b = 0$  and we obtain the discrete Fourier transform (3). In the general case of the non-Abelian groups, we define spin network states  $|\rho, a, b\rangle$  by

$$\prod_e \sqrt{\dim \rho_e} \rho_e(g_e)_{a_e b_e} = \langle g | \rho, a, b \rangle, \quad (58)$$

which are orthonormal; that is,  $\langle \rho, a, b | \rho', a', b' \rangle = \delta_{\rho\rho'} \delta_{aa'} \delta_{bb'}$ .

In the spin network basis, we can easily express the left  $L_e(h)$  and right translation operators  $R_e(h)$ . These are the conjugated operators to the holonomy operators (44). To simplify notation, we will just consider the Hilbert space and states associated to one edge and omit the edge subindex. We define

$$\hat{L}(h) |g\rangle = |h^{-1}g\rangle, \quad \hat{R}(h) |g\rangle = |gh\rangle. \quad (59)$$

In the spin network basis,

$$\begin{aligned} \langle g | \hat{L}(h) | \rho, a, b \rangle &= \langle hg | \rho, a, b \rangle = \sqrt{\dim \rho} \rho(hg)_{ab} \\ &= \sqrt{\dim \rho} \rho(h)_{ac} \rho(g)_{cb} \\ &= \rho(h)_{ac} \langle g | \rho, c, b \rangle, \end{aligned} \quad (60)$$

where we sum over repeated indices. That is,

$$\begin{aligned} \hat{L}(h) | \rho, a, b \rangle &= \rho(h)_{ac} | \rho, c, b \rangle, \\ \hat{R}(h) | \rho, a, b \rangle &= | \rho, a, c \rangle \rho(h^{-1})_{cb}. \end{aligned} \quad (61)$$

The remaining factor  $K_0$  of the transfer operator factorizes over the edges, and it is straightforward to check that it acts on one edge as

$$\hat{K}_0 = \frac{1}{|G|} \sum_h w_{f_i}(h) \hat{L}(h) = \frac{1}{|G|} \sum_h w_{f_i}(h) \hat{R}(h). \quad (62)$$

(Here, one has to use that  $w_{f_i}$  is a class function and invariant under inversion of the argument.) On a spin network state, we obtain

$$\begin{aligned} \hat{K}_0 | \rho, a, b \rangle &= \frac{1}{|G|} \sum_h w_{f_i}(h) \rho(h)_{ac} | \rho, c, b \rangle \\ &= \frac{1}{|G|} \sum_h \sum_{\rho'} c_{\rho'} \chi_{\rho'}(h) \rho(h)_{ac} | \rho, c, b \rangle \\ &= \sum_{\rho} \frac{1}{\dim \rho} c_{\rho} | \rho, a, b \rangle, \end{aligned} \quad (63)$$

where in the second line we used that a class function can be expanded into characters  $\chi_{\rho}$  and in we used the third the orthogonality relation

$$\frac{1}{|G|} \sum_h \chi_{\rho'}(h) \rho(h)_{ac} = \frac{1}{\dim \rho} \delta_{\rho, \rho'} \delta_{ac}. \quad (64)$$

The expansion coefficients  $c_{\rho}$  are given by

$$c_{\rho} = \frac{1}{|G|} \sum_h w_{f_i}(h) \bar{\chi}_{\rho}(h). \quad (65)$$

That is,  $K_0$  acts as a diagonal in the spin network basis, and as the eigenvalues only depend on the representation labels  $\rho$ , it commutes with left and right translations. For the Yang-Mills theory (with Lie groups) in the limit of continuous time, one obtains for  $K_0$  the Laplacian on the group [111, 112].

**6.2. Gauge-Invariant Spin Nets.** Given a spin network function  $\psi$ , which can be regarded as function  $\psi(h_e)$  of holonomies associated to edges, where  $h \in G^{\sharp e}$  is a connection, and a gauge transformation  $\gamma \in G^{\sharp v}$ , an assignment of group elements to vertices of the network, then the gauge transformed spin network function is given by

$$\hat{\Gamma}(\gamma) \psi(h_e) := \psi(\gamma_{s(e)} h_e \gamma_{t(e)}^{-1}), \quad (66)$$

where  $s(e)$  and  $t(e)$  are the source and the target vertices of the edge  $e$ . A gauge transformation can therefore be expressed as a linear operator in terms of  $\hat{L}$  and  $\hat{R}$ , as shown in the last section. Decomposing the spin network function into the basis  $|\rho_e, a_e, b_e\rangle$ , that is,

$$\psi(h_e) = \sum_{\rho_e, a_e, b_e} \tilde{\psi}_{\rho_e, a_e, b_e} | \rho_e, a_e, b_e \rangle, \quad (67)$$

one can readily see that the condition for a function  $\psi$  to be invariant under all gauge transformations  $\alpha_g$  translates into a condition for the coefficients  $\tilde{\psi}_{\rho_e, a_e, b_e}$ ; namely,

$$\tilde{\psi}_{\rho_e, a_e, b_e} = \prod_v \Gamma_{a_{e_1}, \dots, b_{e_n}}^{(v)}, \quad (68)$$

where the product ranges over vertices  $v$ , and each tensor  $t^{(v)}$ , which has the indices  $a_e$  for each edge  $e$  outgoing of and  $b(e)$  for each edge  $e$  incoming  $v$ , is an invariant element of the tensor representation space

$$t^{(v)} \in \text{Inv} \left( \bigotimes_{v=s(e)} V_{\rho_e} \otimes \bigotimes_{v=t(e)} V_{\rho_e}^* \right), \quad (69)$$

that is, an intertwiner between the tensor product of incoming and outgoing representations for each vertex. An orthonormal basis on the space of gauge-invariant spin network functions therefore corresponds to a choice of orthonormal intertwiner in each tensor product representation space for each vertex, where the inner product is the tensor product of the inner products on each  $V_{\rho}, V_{\rho}^*$ .

Note that neither the holonomies  $\widehat{\rho}(h_e)_{ab}$  nor left and right translations are gauge invariant; therefore, they map gauge-invariant functions to nongauge-invariant ones. However, they can be combined into gauge-invariant combinations, such as the holonomy of a Wilson loop within a net.

**6.3. Local Constraint Operators and Physical Inner Product.** We have seen that for a general gauge partition function of the form of (48) the transfer operator (57)

$$\widehat{T} = \widehat{P}_G \widehat{W} \widehat{K}_0 \widehat{W} \widehat{P}_G \quad (70)$$

incorporates a projection  $\widehat{P}_G$  onto the space of gauge-invariant states. This is a general feature of partition functions with gauge symmetries [113, 114]. Indeed in quantum gravity, one attempts to construct a projector onto gauge-invariant states by constructing an appropriate partition function.

As has been discussed in Section 4.1, the *BF*-models enjoy a further gauge symmetry, the translation symmetries (21). (A generalization of these symmetries holds also for the non-Abelian groups.) Hence, we can expect that another projector will appear in the transfer operator. Indeed, it will turn out that the transfer operator for the *BF*-models is just a combination of projectors.

This is straightforward to see as for the *BF*-models that the plaquette weights are given by  $w_f(h) = \delta_G(h)$ , so that

$$\widehat{W} = \prod_{f_s} (\widehat{\delta}_G(h_{f_s}))^{1/2}. \quad (71)$$

Here, one might be worried by taking the square roots of the delta functions; however, we are on a finite group. The operator  $\widehat{W}^2$  is diagonal in the basis  $|g\rangle$  and has only two eigenvalues:  $q^{\#f_s}$  on states where all plaquette holonomies vanish and zero otherwise. The square root of  $\widehat{W}^2$  is defined by taking the square root of these eigenvalues.

Hence,  $\widehat{W}$  is proportional to another projector  $\widehat{P}_F$ , which projects onto the states for which all the plaquette holonomies are trivial, that is, states with zero (local) curvature.

The final factor in the transfer operator  $\widehat{T}$  is  $\widehat{K}_0$  which, according to the matrix element of  $\widehat{K}_0$  in (56) (putting  $w_f(h) = \delta_G(h)$ ), is proportional to the identity

$$\widehat{K}_0 = |G|^{\#e_s} \text{Id}_{\mathcal{H}}, \quad (72)$$

where the numerical factor arises due to our normalization convention for the group delta functions, which amounts to  $\delta_G(\text{id}) = |G|$ . Hence, the transfer operator for the *BF*-theory is given by

$$\widehat{T} = |G|^{\#e_s + \#f_s} \widehat{P}_G \widehat{P}_F = |G|^{\#e_s + \#f_s} \widehat{P}_F \widehat{P}_G, \quad (73)$$

and since for the projectors we have  $\widehat{P}_G^N = \widehat{P}_G$ ,  $\widehat{P}_F^N = \widehat{P}_F$ , the partition function simplifies to

$$\begin{aligned} Z &= \text{tr}_{\mathcal{H}} \widehat{T}^N = |G|^{N(\#e_s + \#f_s)} \text{tr}_{\mathcal{H}} \widehat{P}_G \widehat{P}_F \\ &=: |G|^{N(\#e_s + \#f_s)} \dim(\mathcal{H}_{\text{phys}}). \end{aligned} \quad (74)$$

The projection operators in the transfer operator ensure that only the so-called physical states  $\psi \in \mathcal{H}_{\text{phys}}$  are contributing to the partition function  $Z$ . These are states in the image of the projectors (we will call the corresponding subspace  $\mathcal{H}_{\text{phys}}$ ) and can be equivalently described to be annihilated by constraint operators, which we will discuss below.

The objective in canonical approaches to quantum gravity [11, 12] is to characterize the space of physical states according to the constraints given by general relativity, which follow from the local symmetries of the theory. (The quantization of these constraints in a consistent way is highly complicated [116–118].) Indeed, also in general relativity, as well as in any theory which is invariant under reparametrizations of the time parameter, one would expect that the transfer operator (or the path integral) is given by a projector, so that the property  $\widehat{T}^2 \sim \widehat{T}$  would hold.<sup>26</sup> This would also imply a certain notion of discretization independence, namely, the independence of transition amplitudes on the number of time steps used in the discretization; see also the discussion in [38] on the relation between reparametrization symmetry in the time parameter and discretization independence. For a lattice based on triangulations, one can introduce a local notion of evolution, the so-called tent moves [41, 119–121]. These moves evolve just one vertex and the adjacent cells. Local transfer operators can be associated to these tent moves. These would evolve the spatial hypersurface not globally, but locally, and would allow to choose some arbitrary order of the vertices to evolve. In this way, one can generate different slicings of a triangulation as well as different triangulations. The condition that the transfer operator associated to a tent move is given by a projector would then implement an even stronger notion of triangulation independence.

However, reparametrization invariance, which would lead to such transfer operators, is usually broken by discretizations already for one-dimensional toy models. For “coarse graining and renormalization” methods to regain this symmetry, see [36–38, 122]. In the case of gravity, or more generally nontopological field theories, these methods will however lead to nonlocal couplings for the partition functions [122], for which one would have to modify the transfer operator formalism.

Furthermore, one has to construct a physical inner product on the space of the physical states. This process is quite trivial in the case considered here, as we are considering finite groups and all of the projectors are proper projectors.

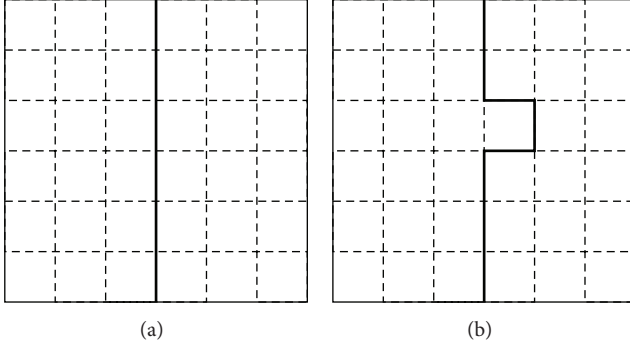


FIGURE 4: Two states in the spin network basis for  $\mathbb{Z}_2$  which are equivalent under the physical inner product. The thick edges carry nontrivial representations. The right state can be obtained from the left state by applying a plaquette stabilizer.

Hence, the physical states are proper states in the “so-called” kinematical Hilbert space  $\mathcal{H}$ , and the inner product of this space can be taken over for the physical states. In contrast, already for the *BF*-theory with “compact” Lie groups, the translation symmetries lead to noncompact orbits. This leads to physical states which are not normalizable with respect to the inner product of the kinematical Hilbert space. For the intricacies of this case (with  $SU(2)$ ), see, for instance, [115]. A further complication for gravity is the very complicated non-commutative structure of the constraints [123–125].

Let us summarize the projectors onto  $\hat{P} = \hat{P}_G \hat{P}_F$ . Also in the case of generalized projectors, a physical inner product can be defined [12, 126] between equivalence classes of kinematical states labelled by representatives  $\psi_1, \psi_2$  through

$$\langle \psi_1 | \psi_2 \rangle_{\text{phys}} := \langle \psi_1 | \hat{P} | \psi_2 \rangle. \quad (75)$$

Kinematical states which project to the same (physical) state define the same equivalence class. This leads, for instance, to an identification of the two states (for the group  $\mathbb{Z}_2$ ) in Figure 4. This is analogous to the action of the spatial diffeomorphism group in “3D and 4D” loop quantum gravity, see the discussion in [115], and reflects the concept of abstract spin foams, whose amplitude does not depend on the embedding into the lattice, in Section 2. Indeed any two states which can be deformed into each other by applying the so-called stabilizer operators introduced below in (76) and (82) are equivalent to each other. The reason is that the physical Hilbert space is the common eigenspace to the eigenvalue 1 with respect to all of the stabilizer operators. Hence, any part of a state that undergoes a nontrivial change if a stabilizer is applied will be projected out in the physical inner product.

Another problem in quantum gravity is then to find “quantum” observables [127–131] that are well defined on the physical Hilbert space. In the case of gravity, these Dirac observables are very hard to obtain explicitly; even classically there are almost none known. In particular, such Dirac observables have to be nonlocal [132]. If one interprets a quantum gravity model as a statistical system, a related task

is to find a well-defined order parameter characterizing the phases. Expectation values of gauge-variant observables may be projected to zero under the physical inner product [133]: As in our case, we work with finite-dimensional Hilbert spaces, and the physical Hilbert space is just a subspace of the kinematical one; there is a construction principle for physical observables. For any operator  $\hat{O}$  on the kinematical Hilbert space, one can consider  $\hat{P}\hat{O}\hat{P}$ , which preserves the physical Hilbert space and corresponds to a “gauge averaging” of  $\hat{O}$ . However, many observables constructed in this way will turn out to be constants. For the *BF*-theory, observables can be constructed by considering a product of the stabilizer operators discussed below along non-contractable loops [54, 134–136]. The resulting observables are nonlocal, and the cardinality of a set of independent operators (equivalently the dimension of the physical Hilbert space) just depends on the topology of the lattice and not on its size.

We will now discuss the stabilizer operators which characterize the physical states. In topological quantum computing, these are known as stabilizer conditions [54].

We have considered the operators  $\hat{\Gamma}(\gamma)$  in (53) that implement a gauge transformation according to the assignments  $(\gamma)_{v_s}$  of group elements to the vertices of the “spatial sub-” lattice. These can be easily localized to the vertices  $v_s$  of the spatial lattice by choosing the gauge parameters  $\gamma$  (an assignment of one group element to every vertex in the spatial lattice) such that all group elements are trivial except for one vertex  $v_s$ . We will call the corresponding operators  $\Gamma_{v_s}(\gamma)$  where now  $\gamma$  denotes an element in the gauge group  $G$  (and not in the direct product  $G^{\#v_s}$ ). These can be expressed by the left and right translation operators (59)

$$\hat{\Gamma}_{v_s}(\gamma) |g\rangle = \prod_{e_s: s(e_s)=v_s} \hat{L}_{e_s}(\gamma) \prod_{e'_s: t(e'_s)=v_s} \hat{R}_{e'_s}(\gamma) |g\rangle. \quad (76)$$

Physical states  $|\psi\rangle$  have to satisfy<sup>27</sup>

$$\hat{\Gamma}_{v_s}(\gamma) |\psi\rangle = |\psi\rangle \quad (77)$$

for all vertices  $v_s$  and all group elements  $\gamma$ . As the operators  $\hat{\Gamma}$  define a representation of the group, we can restrict to the elements of a generating set of the group.

This suggests considering the action of these “star” operators in the spin network basis:

$$\begin{aligned} \hat{\Gamma}_{v_s}(\gamma) |\rho, a, b\rangle &= \prod_{e: s(e)=v_s} \rho_e(\gamma)_{a_{e_e}} \prod_{e': t(e')=v_s} \rho_{e'}(\gamma^{-1})_{d_{e'} b_{e'}} \\ &\times |\rho, (c_e, a_{e'}, a_{e''}), (b_e, d_{e'}, b_{e''})\rangle, \end{aligned} \quad (78)$$

where on the right-hand side edges  $e$  are the ones starting at  $v_s$ ,  $e'$  are those ending at  $v_s$  and  $e''$  are all of the remaining edges in the spatial lattice. From this expression, one can conclude that the spin network states transform at  $v_s$  in a tensor product of representations

$$\rho_{v_s} = \bigotimes_{e: s(e)=v_s} \rho_e \otimes \bigotimes_{e': t(e')=v_s} \rho_{e'}^*, \quad (79)$$

where  $\rho^*$  is the dual representation to  $\rho$ . Gauge-invariant states can be constructed by considering the projections of these representations to the trivial ones, or equivalently by contracting the matrix indices of the states with the intertwiners between the tensor product of “incoming” representations and the tensor product of “outgoing” representations at the vertices; see Section 6.2.

For the Abelian groups  $\mathbb{Z}_q$ , the irreducible representations are one-dimensional; hence, we can omit the indices  $a, b$  in the spin network basis. The tensor product of representations (79) is also one dimensional and equal to the trivial representation provided that

$$\sum_{e: s(e)=v_s} k_e = \sum_{e': t(e')=v_s} k_{e'} \quad (80)$$

for the representation labels  $k_e, k_{e'}$  of the outgoing and incoming edges at  $v_s$ , respectively. Hence, we recover the Gauss constraints from the spin foam representation (12). Here, these Gauss constraints appear as based on vertices as opposed to edges as in (12). But the spatial vertices  $v_s$  represent the timelike edges, and the Gauss constraints in the canonical formalism are just the Gauss constraints associated to the timelike edges in the spin foam representation (12).

Let us turn to the other projector  $\hat{P}_F$ . It maps to states for which all of the spatial plaquettes  $f_s$  have trivial holonomies. Hence, the conditions on physical states can be written as

$$\hat{F}_{f_s, a, b}^\rho |\psi\rangle = \delta_{ab} |\psi\rangle, \quad (81)$$

where we have introduced the plaquette operators (corresponding to the flatness constraints)

$$\hat{F}_{f_s, a, b}^\rho = \left( \prod_{e \in f_s} \hat{\rho} \left( g_e^{o(f_s, e)} \right) \right)_{ab}. \quad (82)$$

Here,  $\rho$  should be a faithful representation; otherwise, one might find that also states with local non-trivial holonomies satisfy the conditions (81); see, for instance, the discussion in [137]. On the other hand, this can be taken as one possible generalization of the model. For the non-Abelian groups, the plaquette operators additionally depend on the choice of a vertex adjacent to the face, at which the holonomy around the face starts and ends. However, the conditions (81) do not depend on this choice of vertex: if a holonomy around a face is trivial for one choice of vertex, then it will be trivial for all other vertices in this face, as these holonomies just differ by a conjugation.

The  $BF$ -theory in any dimension is a topological field theory; that is, there are only finitely many physical degrees of freedom depending on the topology of space. Consequently the number of physical states, or equivalently of equivalence classes of kinematical states in the sense of (75), is finite and does not scale with the lattice volume.

There are a number of generalizations one could consider. First, one can couple matter in the form of defects to the models. In 3D, the  $SU(2)$   $BF$ -theory corresponds to a first-order formulation of 3D gravity. Point-like particles can be coupled to the model, see, for instance, [134–136], and lead

to the violation of the flatness constraint (82) (coupling to the mass of the particles) and to the violation of the Gauss constraints (77) (coupling to the spin of the particles) at the position of the particle. The defects can be understood as changing the topology of the spatial lattice, that is, changing its first fundamental group. To have the same effect in, say, four dimensions, one needs to couple strings; see [138–140].

In the context of quantum computing [54], one does not necessarily impose the conditions (77) and (82) as constraints but as characterizations for the ground states of the system. Elementary excitations or quasi-particles are states in which either the flatness or the Gauss constraints are violated. These excitations appear in pairs (at least for the Abelian models [141]) and can be created by so-called ribbon operators [54, 141–143], which in the context of the proper particle models in 3D gravity are related to gauge-invariant “Dirac” observables [144]. For further generalizations based on symmetry breaking from  $G$  to a subgroup of  $G$ , see, for instance, [141].

In 4D gravity is not topological, rather there are propagating degrees of freedom. Similar to the discussion for the partition functions in Section 4.1, one can try to construct new models which are nearer to gravity by breaking down the symmetries of the  $BF$ -theory. In 3D, the flatness constraints are defined on the plaquettes of the lattice. Constraints act also as generators of gauge transformations (indeed the conditions (77) and (82) impose that physical states have to be gauge invariant), and the flatness constraints can be interpreted as translating the vertices of the dual lattice in space time [39, 40, 145, 146], which can be seen as an action of a diffeomorphism. In 4D, the same interpretation holds only for some exceptional cases corresponding to special lattices that do not lead to space time curvature; see the discussion in [107]. In general, the flatness constraints rather generate translations of the edges of the dual lattice [147].

A possible generalization leading to gravity-like models is therefore to replace the flatness conditions (81) based on plaquettes with some contractions of these conditions, such that these new conditions are based on the 3-cells, that is, cubes in a hyper-cubical lattice. This would correspond to the contractions of the form  $FE$  (the Hamiltonian constraints) and  $FE$  (diffeomorphism constraints) in the “complex” Ashtekar variables of the curvature  $F$  with the electric flux variables  $E$  [11, 12, 148–150]. The difficulty in constructing such models is consistency of the constraint algebra, that is, to find constraints that form a closed algebra. However to consider such models for finite groups should be much simpler than in the case of full gravity. Even if such consistent constraint algebras cannot be found, the physical Hilbert spaces could be constructed, for instance, with the techniques in [123–125, 151].

Furthermore, it will be illuminating to derive the transfer operators for the constrained models introduced in Section 5.3; as in the full theory, the relation between the covariant models and the Hamiltonians in the canonical formalism is still open [152, 153]. To this end, a connection representation [91, 109] can be employed.



## 7. Tensor Models and Tensor Group Field Theories

Tensor models are theories of random topological spaces in the fashion of matrix models [154, 155], of which they may be seen as a superset.

Matrix models are a well-studied theory that can describe a multitude of different scenarios including  $2d$  gravity with cosmological constant [156–159] (optionally coupled to the  $2d$  Ising/Potts matter [160, 161] or the  $2d$  Yang-Mills matters theories [162]), certain string theories [163], the enumeration of virtual knots and links [164–167], and the list goes on. Moreover, they have inspired the development of a plethora of useful techniques to solve and analyse statistical ensembles of random matrices [168] such as the topological expansion [169–172], the eigenvalue method [173], the double-scaling limit, the method of orthogonal polynomials [174], and the character expansion [175, 176], to name just a few. Tensor models are a natural extension of this idea to higher dimensions. The ambition is to develop a similar technology, with similar applications, for tensor models in general.

Despite some initial pessimism [177], a recent spike of optimism occurred within the growing tensor model community, when it was discovered that a large class of such theories [18, 178–182] possessed a  $1/N$ -expansion [183–189]; that is, their Feynman graphs could be partitioned into manageable subsets, using some parameter  $N$ . In fact, it was shown that the leading order sector, in the large- $N$  limit, contained an infinite but manageable number of the Feynman graphs with the topology of the  $D$ -sphere ( $D$  being the rank of the tensor). This leading order sector was analysed for a variety of models, which included the plain model [190, 191], placing the Ising/Potts [192] and dimer [193, 194] matter interactions on these  $D$ -dimensional topological spaces, as well as dually weighing the spaces [195]. Critical exponents were extracted, indicating that this sector of the theory displayed identical physical properties to those of the branched polymer phase of the Euclidean dynamical triangulations [196, 197]. This likelihood was further confirmed when it was shown that their respective Hausdorff and spectral dimensions coincided [198]. Interestingly, aspects of universality have also been displayed by this leading order sector [199], while the quantum symmetries have been catalogued at all orders and take the suggestive form of a Virasoro-like algebra [200–202].

While the majority of the results stated above has been explicitly detailed for the simplest class of tensor models, the independent identically distributed IID tensor models, the initial result on the  $1/N$ -expansion applies to many more classes of tensor models, including certain topological and quantum-gravity-inspired tensor models. Thus, a current aim is to develop the above formalism for these broader tensor model scenarios. As stated in the introduction, spin foams for quantum gravity generically involved the Lie groups, and so more structure, than that provided by tensor models, is needed to describe them.

Tensor group field theories (TGFTs) [19–21, 203] are to quantum field theory as tensor models are to quantum

mechanics. Spin foam diagrams naturally arise as the Feynman graphs of TGFTs [1–5], and indeed various quantum-gravity-inspired TGFTs exist in the literature, such as the Boulatov-Ooguri theories [204, 205], the EPRL and FK theories [22–24, 206], and the Baratin-Oriti theories [25, 26]. As a result, they have garnered increasing attention from the quantum gravity community, since, inherently, all current spin foam models for 4d quantum gravity involve a truncation of the degrees of freedom (due to the use of a single lattice structure) and a manifest loss of diffeomorphism symmetry. With their explicit summation over  $D$ -dimensional topological spaces, the hope is that TGFTs recapture these lost degrees of freedom. Indeed, some evidence has already been found at the level of the TGFT action for a seed of diffeomorphism symmetry [207]. Moreover, many techniques may be applied to these field theories that are idiosyncratic to quantum field theories (as opposed to quantum mechanics). Perhaps the most striking is the study of their renormalisation group properties [208–218], but also work has commenced on the study of mean field theory properties [219], matter coupling [220–222], various symmetry analyses [223, 224], and instantonic field theory solutions [225–228] (including cosmological applications [229]).

Having said all that, the aim of this paper is not to detail spin foam models with the Lie groups, but rather spin foam models with finite groups. This places us back under the purview of tensor models, and it is these that we will describe in the coming section.

To commence, let us detail some of the general setup.

**7.1. General Setup.** Let us construct the class of *independent identically distributed* (IID) models. We will attempt to be precise without being especially detailed. We refer the reader to [230] for a more thorough explanation. The fundamental variable is a complex rank- $D$  tensor ( $D \geq 2$ ), which may be viewed as a map  $T : H_1 \times \cdots \times H_D \rightarrow \mathbb{C}$ , where the  $H_i$  are complex vector spaces of dimension  $N_i$ . It is a tensor, so it transforms covariantly under a change of basis of each vector space independently. Its complex-conjugate  $\bar{T}$  is its contravariant counterpart.

One refers to their components in a given basis by  $T_g$  and  $\bar{T}_{\bar{g}}$ , where  $g = \{g_1, \dots, g_D\}$ ,  $\bar{g} = \{\bar{g}_1, \dots, \bar{g}_D\}$ , and the bar ( $-$ ) distinguishes contravariant from covariant indices. We have purposefully denoted the indices by  $g_i$  as they may be viewed as elements of respective  $\mathbb{Z}_{N_i}$ .

As one might imagine, with these ingredients, one can build objects that are invariant under changes of bases. These so-called *trace invariants* are a subset of  $(T, \bar{T})$ -dependent monomials that are built by pairwise contracting covariant and contravariant indices until all indices are saturated. It emerges readily that the pattern of contractions for a given trace invariant is associated to a unique *closed  $D$ -colored graph*, in the sense that, given such a graph, one can reconstruct the corresponding trace invariant and vice versa.<sup>28</sup>

In Figure 5, we illustrate the graph  $\mathcal{B}_1$ , the unique closed  $D$ -colored graph with two vertices, which represents the unique quadratic trace invariant  $\text{tr}_{\mathcal{B}_1}(T, \bar{T}) = T_g \delta_{g\bar{g}} \bar{T}_{\bar{g}}$ .

More generally, we denote the trace invariant corresponding to the graph  $\mathcal{B}$  by  $\text{tr}_{\mathcal{B}}(T, \bar{T})$ .

From now on, we will make two restrictions: (i) all of the vector spaces have the same dimension  $N$  and (ii) we consider only *connected* trace invariants, that is, trace invariants corresponding to graphs with just a single connected component.

Given these provisos, the most general invariant action for such tensors is

$$S(T, \bar{T}) = \text{tr}_{\mathcal{B}_1}(T, \bar{T}) + \sum_{k=2}^{\infty} \sum_{\mathcal{B} \in \Gamma_k^{(D)}} \frac{t_{\mathcal{B}}}{N^{(2/(D-2)!) \omega(\mathcal{B})}} \text{tr}_{\mathcal{B}}(T, \bar{T}), \quad (83)$$

where  $\Gamma_k^{(D)}$  is the set of connected closed  $D$ -colored graphs with  $2k$  vertices,  $\{t_{\mathcal{B}}\}$  is the set of coupling constants, and  $\omega(\mathcal{B}) \geq 0$  is the *degree* of  $\mathcal{B}$  (see [18] for its definition and properties). This defines the IID class of models.

**7.2.  $1/N$ -Expansion.** The central objects for further investigation are the *free energy* (per degree of freedom) associated to these models:

$$E(\{t_{\mathcal{B}}\}) = \frac{1}{N^D} \log \left( \int dT d\bar{T} e^{-N^{D-1} S(T, \bar{T})} \right), \quad (84)$$

along with the various other  $n$ -point Green functions. When facing such a quantity, the standard procedure is to expand it in a Taylor series with respect to the coupling constants  $t_{\mathcal{B}}$  and to evaluate the resulting Gaussian integrals in terms of the Wick contractions. It transpires that the Feynman graphs  $\mathcal{G}$  contributing to  $E(\{t_{\mathcal{B}}\})$  are none other than *connected closed*  $(D+1)$ -colored graphs with weight:

$$A_{\mathcal{G}} = \frac{(-1)^{|\rho|}}{\text{SYM}(\mathcal{G})} \left( \prod_{\rho} t_{\mathcal{B}_{(\rho)}} \right) N^{-(2/(D-1)!) \omega(\mathcal{G})}, \quad (85)$$

where

$$\omega(\mathcal{G}) = \frac{(D-1)!}{2} \left[ D + \frac{D(D-1)}{4} |\mathcal{V}_{\mathcal{G}}| - |\mathcal{F}_{\mathcal{G}}| \right], \quad (86)$$

$\text{SYM}(\mathcal{G})$  is a symmetry factor, and  $\mathcal{B}_{(\rho)}$  runs over the subgraphs with colors  $\{1, \dots, D\}$  of  $\mathcal{G}$ , while  $\mathcal{V}_{\mathcal{G}}$  and  $\mathcal{F}_{\mathcal{G}}$  are the vertex and face sets of  $\mathcal{G}$ , respectively. For  $D = 2$ , the *degree*  $\omega(\mathcal{G})$  coincides with the genus of a surface specified by  $\mathcal{G}$ . A few more words of explanation are most definitely in order here. The graphs  $\mathcal{G} \in \Gamma^{(D+1)}$  arise in the following manner. One knows that a given term in the Taylor expansion is a product of trace invariants upon which one performs Wick contractions. For such a term, one indexes these trace invariants by  $\rho \in \{1, \dots, \rho^{\max}\}$ ; that is, we index their associated  $D$ -colored graphs  $\mathcal{B}_{(\rho)}$ . A single Wick contraction pairs a tensor  $T$ , lying somewhere in the product, with a tensor  $\bar{T}$  lying somewhere else. One represents such a contraction by joining the black vertex representing  $T$  to the white vertex representing  $\bar{T}$  with a line of color 0. Thus, a complete set of the Wick contractions results in a connected (as one is dealing with the free energy) closed  $(D+1)$ -colored graph. A particular Wick contraction is drawn in Figure 6.

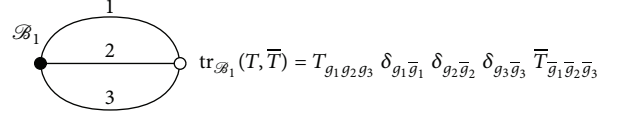


FIGURE 5: A closed  $D$ -colored graph and its associated trace invariant (for  $D = 3$ ).

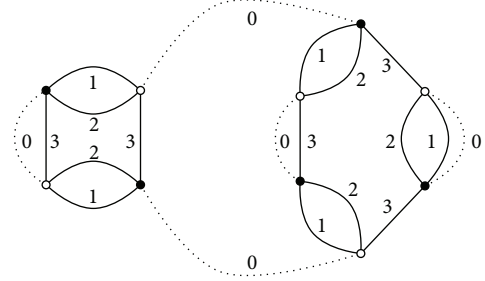


FIGURE 6: A Wick contraction of two trace invariants (for  $D = 3$ ). The contraction of each  $(T, \bar{T})$  pair is represented by a dashed line of color 0.

It requires a bit more work to reconstruct the amplitude explicitly; see [230]. Importantly,  $\omega(\mathcal{G})$  is a nonnegative integer, and so one can order the terms in the Taylor expansion of (84) according to their power of  $1/N$ . Quite evidently, therefore, one has a  $1/N$ -*expansion*.

**7.3. Interpretation.** Strikingly,  $(D+1)$ -colored graphs represent  $D$ -dimensional simplicial pseudomanifolds. We will give a rough presentation here. One distinguishes the  $k$ -bubbles of species  $\{i_1, \dots, i_k\}$  as those maximally connected subgraphs with the colors  $\{i_1, \dots, i_k\}$ . These  $k$ -bubbles are identified with the  $(D-k)$ -simplices of the associated simplicial complexes. Moreover, one can see clearly that the  $k$ -bubbles of species  $\{i_1, \dots, i_k\}$  are nested within  $(k+1)$ -bubbles of species  $\{i_1, \dots, i_k, j\}$ , where  $j \in \{0, \dots, D\} \setminus \{i_1, \dots, i_k\}$ . This set of nested relationships encodes the gluing of the simplices within the simplicial complex. This argument may be made rigorously. Thus, at the very least, rank- $D$  tensor models capture a sum over  $D$ -dimensional topological spaces.

Moreover, a geometrical interpretation may be attached most readily to the amplitudes of the IID model by setting  $t_{\mathcal{B}} = g^{|\mathcal{V}_{\mathcal{B}}|/2} / \text{SYM}(\mathcal{B})$  for all  $\mathcal{B}$ , where  $g$  is some coupling constant. Then, one may rewrite the amplitudes as

$$\text{SYM}(\mathcal{G}) A_{\mathcal{G}} = e^{\kappa_{D-2} \mathcal{N}_{D-2} - \kappa_D \mathcal{N}_D}, \quad (87)$$

where  $\mathcal{N}_k$  denotes the number of  $k$  simplices in the simplicial complex associated to  $\mathcal{G}$  and

$$\kappa_{D-2} = \log N, \quad \kappa_D = \frac{1}{2} \left( \frac{1}{2} D(D-1) \log N - \log g \right). \quad (88)$$

Interestingly, the discretization of the Einstein-Hilbert action on an equilateral  $D$ -dimensional simplicial complex takes the form

$$\begin{aligned} S_{\text{EDT}} &= \Lambda \sum_{\sigma_D} \text{vol}(\sigma_D) - \frac{1}{16\pi G_{\text{Newton}}} \\ &\quad \times \sum_{\sigma_{D-2}} \text{vol}(\sigma_{D-2}) \delta(\sigma_{D-2}) \\ &= \Lambda \text{vol}(\sigma_D) \mathcal{N}_D - \frac{\text{vol}(\sigma_{D-2})}{16\pi G_{\text{Newton}}} \\ &\quad \times \left( 2\pi \mathcal{N}_{D-2} - \frac{D(D+1)}{2} \arccos\left(\frac{1}{D}\right) \mathcal{N}_D \right), \end{aligned} \quad (89)$$

where  $G_{\text{Newton}}$  and  $\Lambda$  are the bare Newton and cosmological constants, respectively, and  $\text{vol}(\sigma_k) = (a^k/k!) \sqrt{(k+1)/2^k}$  is the volume of the equilateral  $k$ -simplex, while  $\delta(\sigma_{D-2})$  denotes the deficit angles associated to the  $(D-2)$ -simplices. If one further associates  $(N, g)$  to  $(G_{\text{Newton}}, \Lambda)$  in the following fashion:

$$\begin{aligned} \log N &= \frac{\text{vol}(\sigma_{D-2})}{8G_{\text{Newton}}} \log g \\ &= \frac{D}{16\pi G_{\text{Newton}}} \text{vol}(\sigma_{D-2}) \\ &\quad \times \left( \pi(D-1) - (D+1) \arccos\left(\frac{1}{D}\right) \right) \\ &\quad - 2\Lambda \text{vol}(\sigma_D) \\ &:= -2a^D \widetilde{\Lambda}, \end{aligned} \quad (90)$$

then one finds that the Feynman amplitudes for the IID model are coincide with those in a Euclidean dynamical triangulations approach. We will return to this later.

**7.4. Large- $N$  Limit.** In the large- $N$  limit, only one subclass of graphs survives, containing those graphs with  $\omega(\mathcal{G}) = 0$ . At this point, there is a marked difference between two and higher dimensions.

- (i) In the 2-dimensional model,  $\omega(\mathcal{G})$  is the genus of the graph in question. Thus, the graphs surviving in this limit are all graphs with the topology of the 2-sphere.
- (ii) In higher dimensions, that is  $D \geq 3$ , it was shown in [190, 191] that the only graphs surviving this limiting procedure are the melonic graphs (with this name stemming from their distinctive structure). While these have the topology of the  $D$ -sphere, they do not constitute all possible  $(D+1)$ -colored graphs with this topology.

The series constituting the leading order sector

$$E_{\text{LO}}(\{t_{\mathcal{B}}\}) = \sum_{\mathcal{G}: \omega(\mathcal{G})=0} \frac{(-1)^{|\rho|}}{\text{SYM}(\mathcal{G})} \prod_{\rho} t_{\mathcal{B}(\rho)} \quad (91)$$

has a finite radius of convergence. This indicates that the theory displays critical behaviour, for some values of the coupling constants, characterised by some critical exponents. There are various scenarios that one might investigate. One simple, yet interesting case is to set  $t_{\mathcal{B}} = g^{|\mathcal{B}|/2} / \text{SYM}(\mathcal{B})$  for all  $\mathcal{B}$ , where  $g$  is some coupling constant. In this scenario, the series display the following critical behaviour:

$$E_{\text{LO}}(g) \sim \left(1 - \frac{g}{g_c}\right)^{2-\gamma}, \quad (92)$$

where

$$\begin{aligned} g_c &= -\frac{1}{48}, \quad \gamma = -\frac{1}{2}, \quad \text{for } D = 2, \\ g_c &= \frac{D^D}{(D+1)^{D+1}}, \quad \gamma = \frac{1}{2}, \quad \text{for } D \geq 3. \end{aligned} \quad (93)$$

Multicritical behaviour can be extracted by tuning several coupling constants independently.

Given the description provided in the previous subsection, one notices that the large- $N$  limit corresponds to  $G_{\text{Newton}} \rightarrow 0$ . Moreover, tuning  $g \rightarrow g_c$ , one enters the regime dominated by graphs with large numbers of vertices (or equivalently, simplicial complexes with large numbers of simplices). As it stands, this corresponds to a large-volume limit. However, with some more work, one can interpret it as a continuum limit. To begin, the average volume is

$$\begin{aligned} \langle \text{Volume} \rangle &= a^D \langle \mathcal{N}_D \rangle \\ &= 2a^D g \frac{\partial}{\partial g} \log E_{\text{LO}}(g) \\ &\sim a^D \left(1 - \frac{g}{g_c}\right)^{-1} \\ &= \frac{\widetilde{\Lambda}}{\log g} \left(1 - \frac{g}{g_c}\right)^{-1}. \end{aligned} \quad (94)$$

To obtain a continuum limit, one should tune  $g \rightarrow g_c$  while keeping the average volume finite. This may be achieved in the following fashion:

$$g \rightarrow g_c, \quad a \rightarrow 0, \quad (95)$$

while keeping

$$\widetilde{\Lambda}_R = -\frac{\widetilde{\Lambda}}{\log g} \left(1 - \frac{g}{g_c}\right) \text{ fixed}, \quad (96)$$

where  $\widetilde{\Lambda}_R$  is a renormalized cosmological constant.

**7.5. Coupling to the Ising and Potts Matter.** The coupling to the Ising/Potts matter in tensor models is a direct generalisation of that scenario in matrix models [231, 232]. For the Ising

matter, one considers a model with two complex tensors and action:

$$\begin{aligned}
 S(T^1, T^2, \bar{T}^1, \bar{T}^2) \\
 = C_{ij} \text{tr}_{\mathcal{B}_1} \left( T^i, \bar{T}^j \right) + \sum_{i=1}^2 \sum_{k=2}^{\infty} \sum_{\mathcal{B} \in \Gamma_k^{(D)}} \frac{t_{\mathcal{B}}}{N^{(2/(D-2)!) \omega(\mathcal{B})}} \text{tr}_{\mathcal{B}} \\
 \times (T^i, \bar{T}^j),
 \end{aligned} \tag{97}$$

where

$$C_{ij} = \begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix} \tag{98}$$

and  $c$  is a coupling constant. This class of models has been examined in [192], where critical exponents have been extracted, which coincide with those of branched polymers.

This matter model may be extended to the  $q$ -state Potts matter by increasing the number of tensors along with a suitable alteration of the propagator.

**7.6. Non-IID Models.** The IID models are but the simplest of a much larger class of models possessing a  $1/N$ -expansion, defined by actions of the form

$$S(T, \bar{T}) = T_g K_{g, \bar{g}} \bar{T}_{\bar{g}} + \sum_{k=2}^{\infty} \sum_{\mathcal{B} \in \Gamma_k^{(D)}} \frac{t_{\mathcal{B}}}{N^{\alpha(\mathcal{B})}} \text{tr}_{\mathcal{B}}(T, \bar{T}). \tag{99}$$

The difference resides in the propagator and the scalings,  $\alpha(\mathcal{B})$ , of the coupling constant, which can be chosen to reproduce any local (quantum-gravity-inspired) spin foam model (in this case, with the finite rather than the Lie group information). As an example, let us briefly detail the choice that yields the Boulatov-Ooguri class of tensor models. Firstly, the propagator is chosen to be

$$K_{g, \bar{g}} = \sum_{h \in \mathbb{Z}_N} \prod_{i=0}^D \delta_{g_i h \bar{g}_i^{-1}}. \tag{100}$$

The  $\alpha(\mathcal{B})$  can be specified so that the resulting amplitudes for the free energy take the form

$$E(\{t_{\mathcal{B}}\}) = \sum_{\mathcal{G}} \frac{(-1)^{|\rho|}}{\text{SYM}(\mathcal{G})} \prod_{\rho} t_{\mathcal{B}(\rho)} \left[ \prod_{e \in \mathcal{E}_{\mathcal{G}}} \sum_{h_e} \prod_{f \in \mathcal{F}_{\mathcal{G}}} \delta_{H_f} \right], \tag{101}$$

where

$$H_f = \prod_{e \in \partial f} h_e^{o(e, f)}. \tag{102}$$

Given that  $\mathcal{E}_{\mathcal{G}}$  and  $\mathcal{F}_{\mathcal{G}}$  are the edge and face sets of  $\mathcal{G}$ , respectively, while  $o(e, f)$  is the respective orientation of the edge  $e$  lying in the boundary of the face  $f$ , it is clear that  $H_f$  is the holonomy around the face  $f$ . As a result, the amplitude within square brackets is the  $BF$ -theory amplitude associated

to the topological manifold represented by  $\mathcal{G}$ . It may be evaluated to obtain some  $\mathcal{G}$ -dependent factor of  $N$  (actually directly dependent on the second Betti number of  $\mathcal{G}$  [233–235]), which allows the graphs to be organised into a  $1/N$ -expansion.

A more in-depth analysis has yet to be completed, although these preliminary results are very promising. One may modify the propagator further to obtain the EPRL, FK [22–24], and BO [25, 26] quantum gravity spin foam amplitudes.

## 8. Nonlocal Spin Foams

We now turn briefly to one of the main open problems facing quantum gravity practitioners: the study of the large-scale behaviour emerging from various models. We will restrict ourselves here to two points: the first motivates the use of the simple models considered here in this endeavour; the second comments on how the study of large-scale behaviour may necessitate the treatment of nonlocality.

To pass from small to large scales, it is clear that renormalization group methods could be useful. However, the application of such techniques, at least to those spin foam models currently discussed as quantum gravity candidates [57–61], is hindered not only by many conceptual issues but also by the tremendously complicated amplitudes emerging from these models. Here, our “toy spin foam models” could help, both to develop new techniques and to investigate the statistical field theory aspects of the models in question.

As a matter of fact, it may be the case that certain properties are independent of the specifics of the amplitudes. For instance, consider the questions of whether and how to sum over topologies within spin foam models. This might not depend on the chosen group underlying the model.

On top of that, note that there are a number of quantum gravity approaches which rather work with quite simple (vertex) amplitudes [13–17, 73–76] (we have in mind here the work of [16, 17] in particular), in which the question of the large-scale limit can be addressed. The models proposed in our work here can be seen as small spin, cut-off versions of the full theory. The hope is that for these models one can replicate the successes of [16, 17] by probing the many-particle (and small-spin) regime, in contrast to the very few-particle and large-spin regime considered up to now.

There has been some very interesting work exploring coarse graining in the spin foam context [236–238]. Independently, the related concept of tensor networks [239–242], a generalization of the spin nets introduced here, has been developed as a tool for coarse graining. In most frameworks, the local form of the spin foams does not change. It may be necessary to accommodate also for nonlocal couplings,<sup>29</sup> in particular if one attempts to regain diffeomorphism symmetry, which is broken by the discretizations employed in many quantum gravity models [36, 37, 122]. As explained in [243], such nonlocal couplings can be accommodated into the tensor network renormalization framework. In this case, the nonlocal couplings are compensated by introducing more



and more complicated building blocks (carrying higher-order variables).

Indeed, after applying block spin transformations to a nontopological lattice gauge system, one can expect to possess a partition function of the form

$$Z = \sum_{\{g\}} \prod_{M=1}^{\infty} \prod_{l_1, \dots, l_M} w_{l_1, \dots, l_M}(h_{l_1}, \dots, h_{l_M}), \quad (103)$$

where  $h_{l_i}$  are holonomies around loops (that is, around plaquettes but also around other surfaces made of several plaquettes) and  $w_{l_1, \dots, l_M}$  is a class function of its arguments, describing possible couplings between (Wilson) loops. A character expansion would introduce representation labels not only on the basic plaquettes but also on all of the other surfaces encircled by loops appearing in (103). The resulting structure is akin to a two-dimensional generalization of a graph. Graphs would appear as effective descriptions for the coarse graining of the Ising-like models discussed in Section 2. Such nonlocal “spin graphs” would be generalizations of the spin nets introduced there. Similarly, graphs have been introduced in [27, 28] to accommodate nonlocal couplings and to describe phase transitions between a geometric phase (i.e., a phase where, for instance, a space time dimension can be defined) and a nongeometric phase. We leave the exploration of the ensuing structures for future work.

## 9. Outlook

In this work, we discussed several concepts and tools which arose in the spin foam, loop quantum gravity, and group field theory approach to quantum gravity and applied these to finite groups. We encountered different classes of theories: one is the well-known example of the Yang-Mills-like theories; others are the topological *BF*-theories. The latter are also well known in condensed matter and quantum computing. A third class of theories are the constrained models discussed in Section 5.3 which mimic the construction of the gravitational models. These are only applicable for the non-Abelian groups as for the Abelian groups the invariant Hilbert space associated to the edges is always one dimensional and cannot be further restricted. We plan to study these theories in more detail in the future, in particular the symmetries and the associated transfer or constraint operators. The relative simplicity of these models (compared with the full theory) allows for the prospect of a complete classification of the choices for the edge projectors and hence of the different constrained models. For the study of the translation symmetries in the non-Abelian models, it might be fruitful to employ a non-commutative Fourier transform [25, 26, 207, 244–246]. This would define an alternative dual for the non-Abelian models, in which the edge projectors carry delta function factors, however defined on a noncommutative space.

We also mentioned the possibilities to obtain gravity-like models by breaking down the translation symmetries of the *BF*-theory. This could be particularly promising in the canonical formalism where one can be guided by the form of

the Hamiltonian constraints for gravity. This strategy is also available for the Abelian groups. In particular for  $\mathbb{Z}_2$ , it is in principle possible to parametrize all possible Hamiltonians or partition functions and to study the associated symmetry content. See also the universality result in [89]. Hence, there might be a definitive answer whether gravity-like models exist, that is, 4D ( $\mathbb{Z}_2$ ) lattice models with translation symmetries based on the 4-cells (or the dual vertices).

Finally, we hope that these models can be helpful in order to develop techniques for coarse graining and renormalization of spin foam models and in group field theories. Here, the connection to standard theories could be exploited, and techniques can be taken over from the known examples and with adjustments being applied to quantum gravity models. To this end, the finite group models could be an important link and provide a class of toy models on which ideas can be tested more easily. For instance, the connection between (real space) coarse graining of spin foams and renormalization in group field theories, which generate spin foams as the Feynman diagrams, could be explored more explicitly than in the (divergent)  $SU(2)$ -based models.

It will be particularly interesting to access the many-particle regime, for instance, using the Monte Carlo simulations, which for the full models is yet out of reach. In the ideal case, it might be possible to explore the phase structure of the constrained theories and to study the symmetry content of these different phases.

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## Endnotes

1. In some cases, the resulting models might then be reformulated as “tiling” models on a fixed lattice [30–32].
2. The small-spin regime arises as the spin is just a label for representation of the group, and for finite groups there is only a finite number of irreducible unitary representations.
3. The models can be extended to oriented graphs. We will however not be interested in the most general situation but will assume that the lattice or more generally cell complex is sufficiently nice in order to construct the dual lattices or complexes. Later on, we will also need to be able to identify higher-dimensional cells (2-cells and 3-cells) in the lattice.
4. The reader may be more familiar with the form of the model in which the variables take the values  $g_v \in \{-1, 1\}$ , which corresponds to the matrix representation of  $\mathbb{Z}_2$ :  $g \rightarrow e^{\pi i g}$ . These are two realizations of the same model, and their partition functions may be related by the following redefinition:

$$Z_{\beta}^{\{-1,1\}} = e^{-\beta} Z_{2\beta}^{\{0,1\}}. \quad (104)$$

5. Note that the factors of  $q$  disappear because

$$\delta^{(q)}(k) = \frac{1}{q} \sum_{g=0}^{q-1} \bar{\chi}_k(g) = \frac{1}{q} \sum_{g=0}^{q-1} \chi_k(g). \quad (105)$$

6. Note that in this particular example we are abusing language twice as we do have neither a “spin” nor a “foam”. The term spin arises in the full models from using the representation labels (spin quantum numbers) of  $SU(2)$ . The proper “foams” arise for lattice gauge theories as a higher-dimensional generalization of the spin nets introduced in this section.
7. Here, we assume that the lattice possesses trivial cohomology; otherwise, one may find the so-called topological models, see [94].
8. This definition is taken over from [83], in which similar objects, known as branched surfaces, were defined for spin foams. We will discuss such objects directly in the following section. In general, such discussions in this paper are motivated by similar considerations for spin foams-like structures appearing in lattice gauge theories [1–6, 83, 84].
9. The free energy is defined with respect to the partition function as

$$F = \log Z. \quad (106)$$

In the perturbative expansion, contributions to the free energy come from single spin nets, whereas the partition function contains contributions from arbitrary products of spin nets embedded into the lattice.

10. For the non-Abelian groups, the branchings or spin foam edges carry intertwiners between the representations associated to the surfaces meeting at the edges.
11. A Wilson loop is a contiguous loop of lattice edges.
12. The orientation of the edges is the one induced by the orientation of the face and  $g_e = g_{e^{-1}}^{-1}$ , where  $e^{-1}$  is the edge with opposite orientation to  $e$ .
13. Indeed, in the zero-temperature limit, the partition function for the Ising gauge models enforces all plaquette holonomies to be trivial. This coincides with the partition function of the BF-theories discussed in Section 4.
14. Here,  $\wedge$  and  $\star$  are the exterior product and the Hodge dual operators on space time forms.
15. As before, the product on  $\mathbb{Z}_q$  is an addition modulo  $q$ :

$$h_f(g_e) = \sum_{e \subset f} o(f, e) g_e. \quad (107)$$

16. However, one should note that not all divergences are due to this gauge symmetry [105, 106].
17. In  $2d$ , this symmetry degenerates into a global symmetry since the Gauss constraints force all  $k_f$  to be equal:  $k_f \equiv k$ . However, the contribution of every representation label  $k$  to the partition function is the same.

18. The gauge parameters are then the Lie algebra-valued fields, and for the Lie algebra of  $SU(2)$ , they truly correspond to translations.
19. More specifically, we consider only complexes which are such that the gluing maps used to successively build up  $\kappa$  are injective. This in particular means that the boundary of each face contains each edge at most once. With certain choices for amplitudes, this condition can be relaxed slightly; see, for example, [85–87].
20. This can be easily shown by proving that  $P_e = P_e^*$ , resulting from the unitarity of all representations, and that  $(P_e)^2 = P_e$ , which uses the translation-invariance and normalization of the Haar measure on  $G$ .
21. It is defined by  $\rho^*(h)_{ab} = \rho(h^{-1})_{ba}$ .
22. Note that there is some ambiguity in the literature about what is actually called the  $6j$ -symbol, which is a question of the correct normalization. We mean the normalized  $6j$ -symbol here, since we chose the intertwiners to be normalized in the first place. The normalization, which usually results in nontrivial edge amplitudes, can be absorbed into the vertex amplitude. Also there might be, according to convention, some sign factors assigned to the edges  $e$ , which may not be absorbable into the vertex amplitudes  $\mathcal{A}_v$ .
23. We are thankful to Frank Hellmann for this insight.
24. See [41, 119–121] for a possibility to introduce a slicing of triangulations into hypersurfaces, and a transfer operator based on the so-called tent moves.
25. All functions have finite norm since we consider finite groups.
26. In the limit of taking the discretization in time direction to the continuum, we would obtain the same projector as for finite time discretization. Indeed, the projectors already encode for finite time steps the “Hamiltonian” constraints, which are the generators of time translations (time reparametrization symmetry).
27. Here,  $\hat{\Gamma}_{v_s}$  is a unitary operator, so that the condition on the physical states is in the form of a so-called stabilizer. Alternatively, the condition can be expressed by self-adjoint constraints  $\widehat{C}_{v_s}(\gamma) = i(\Gamma_{v_s}(\gamma) - \Gamma_{v_s}(\gamma^{-1}))$  for group elements  $\gamma$  with  $\gamma \neq \gamma^{-1}$ . Otherwise, define  $C_{v_s}(\gamma) = \Gamma_{v_s}(\gamma) - \text{Id}_{\mathcal{H}}$ . Physical states are then annihilated by the constraints.
28. While we refer the reader to [18] for various definitions, it is perhaps not unwise to match up right here the defining properties of a closed  $D$ -colored graph  $\mathcal{B}$  with those of a trace invariant: (i)  $\mathcal{B}$  has two types of vertex, labelled black and white, that represent the two types of tensor,  $T$  and  $\bar{T}$ , respectively. (ii) Both types of vertex are  $D$ -valent, which matches the property that both types of tensor have  $D$ -indices. (iii)  $\mathcal{B}$  is bipartite, meaning that black vertices are directly joined only to white vertices and vice versa. This is in correspondence with the fact that indices are contracted in covariant-contravariant pairs. (iv) Every edge of  $\mathcal{B}$  is colored by a single element

of  $\{1, \dots, D\}$ , such that the  $D$ -edges emanating from any given vertex possess distinct colors. This represents the fact that the indices index distinct vector spaces so that a covariant index in the  $i$ th position must be contracted with some contravariant index in the  $i$ th position. (v)  $\mathcal{B}$  is closed, representing that every index is contracted.

29. These couplings are in general exponentially suppressed with respect to some nonlocality parameter, like the distance or size of the Wilson loops. In this case, one would still speak of a local theory.

## References

- [1] M. P. Reisenberger, "World sheet formulations of gauge theories and gravity," <http://arxiv.org/abs/gr-qc/9412035>.
- [2] J. Iwasaki, "A Definition of the Ponzano-Regge quantum gravity model in terms of surfaces," *Journal of Mathematical Physics*, vol. 36, no. 11, p. 6288, 1995.
- [3] M. P. Reisenberger and C. Rovelli, "Sum over surfaces form of loop quantum gravity," *Physical Review D*, vol. 56, no. 6, pp. 3490–3508, 1997.
- [4] M. Reisenberger and C. Rovelli, "Spin foams as Feynman diagrams," <http://arxiv.org/abs/gr-qc/0002083>.
- [5] M. P. Reisenberger and C. Rovelli, "Space-time as a Feynman diagram: the connection formulation," *Classical and Quantum Gravity*, vol. 18, no. 1, pp. 121–140, 2001.
- [6] J. C. Baez, "Spin foam models," *Classical and Quantum Gravity*, vol. 15, no. 7, p. 1827, 1998.
- [7] A. Perez, "Spin foam models for quantum gravity," *Classical and Quantum Gravity*, vol. 20, no. 6, p. R43, 2003.
- [8] A. Perez, "Introduction to loop quantum gravity and spin foams," <http://arxiv.org/abs/gr-qc/0409061>.
- [9] R. Loll, "Discrete approaches to quantum gravity in four dimensions," *Living Reviews in Relativity*, vol. 1, p. 13, 1998.
- [10] F. J. Wegner, "Duality in generalized Ising models and phase transitions without local order parameters," *Journal of Mathematical Physics*, vol. 12, no. 10, pp. 2259–2272, 1971.
- [11] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge, UK, 2004.
- [12] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge University Press, Cambridge, UK, 2005.
- [13] J. Ambjorn, "Quantum gravity represented as dynamical triangulations," *Classical and Quantum Gravity*, vol. 12, no. 9, p. 2079, 1995.
- [14] J. Ambjorn, D. Boulatov, J. L. Nielsen, J. Rolf, and Y. Watabiki, "The spectral dimension of 2D quantum gravity," *Journal of High Energy Physics*, vol. 02, p. 010, 1998.
- [15] J. Ambjorn, B. Durhuus, and T. Jonsson, *Quantum Geometry. A Statistical Field Theory Approach*, Cambridge Monographs in Mathematical Physics, Cambridge University Press, Cambridge, UK, 1997.
- [16] J. Ambjorn, J. Jurkiewicz, and R. Loll, "The universe from scratch," *Contemporary Physics*, vol. 47, no. 2, pp. 103–117, 2006.
- [17] J. Ambjorn, A. Gorlich, J. Jurkiewicz, R. Loll, J. Gizbert-Studnicki, and T. Trzesniewski, "The semiclassical limit of causal dynamical triangulations," *Nuclear Physics B*, vol. 849, no. 1, pp. 144–165, 2011.
- [18] R. Gurau and J. P. Ryan, "Colored tensor models-a review," *SIGMA*, vol. 8, article 020, 78 pages, 2012.
- [19] D. Oriti, "The Group field theory approach to quantum gravity," in *Approaches to Quantum Gravity*, D. Oriti, Ed., pp. 310–331, Cambridge University Press, Cambridge, UK, 2007.
- [20] D. Oriti, "Group field theory as the microscopic description of the quantum spacetime fluid: a new perspective on the continuum in quantum gravity," *PoS QG*, p. 030, 2007.
- [21] L. Freidel, "Group field theory: an Overview," *International Journal of Theoretical Physics*, vol. 44, no. 10, pp. 1769–1783, 2005.
- [22] T. Krajewski, J. Magnen, V. Rivasseau, A. Tanasa, and P. Vitale, "Quantum corrections in the group field theory formulation of the Engle-Pereira-Rovelli-Livine and Freidel-Krasnov models," *Physical Review D*, vol. 82, no. 12, Article ID 124069, 2010.
- [23] J. B. Geloun, R. Gurau, and V. Rivasseau, "EPRL/FK group field theory," *EPL*, vol. 92, no. 6, Article ID 60008, 2010.
- [24] E. R. Livine and D. Oriti, "Coupling of spacetime atoms and spin foam renormalisation from group field theory," *Journal of High Energy Physics*, vol. 0702, p. 092, 2007.
- [25] A. Baratin and D. Oriti, "Quantum simplicial geometry in the group field theory formalism: reconsidering the Barrett-Crane model," *New Journal of Physics*, vol. 13, Article ID 125011, 2011.
- [26] A. Baratin and D. Oriti, "Group field theory and simplicial gravity path integrals: a model for Holst-Plebanski gravity," *Physical Review D*, vol. 85, no. 4, Article ID 044003, 2012.
- [27] T. Konopka, F. Markopoulou, and L. Smolin, "Quantum graphity," <http://arxiv.org/abs/hep-th/0611197>.
- [28] T. Konopka, F. Markopoulou, and S. Severini, "Quantum graphity: a model of emergent locality," *Physical Review D*, vol. 77, no. 10, Article ID 104029, 2008.
- [29] D. Benedetti and J. Henson, "Imposing causality on a matrix model," *Physics Letters B*, vol. 678, no. 2, pp. 222–226, 2009.
- [30] B. Dittrich and R. Loll, "A hexagon model for 3D Lorentzian quantum cosmology," *Physical Review D*, vol. 66, no. 8, Article ID 084016, 2002.
- [31] B. Dittrich and R. Loll, "Counting a black hole in Lorentzian product triangulations," *Classical and Quantum Gravity*, vol. 23, no. 11, Article ID 3849, pp. 3849–3878, 2006.
- [32] D. Benedetti, R. Loll, and F. Zamponi, "(2+1)-dimensional quantum gravity as the continuum limit of causal dynamical triangulations," *Physical Review D*, vol. 76, no. 10, Article ID 104022, 2007.
- [33] P. Hasenfratz and F. Niedermayer, "Perfect lattice action for asymptotically free theories," *Nuclear Physics B*, vol. 414, no. 3, pp. 785–814, 1994.
- [34] P. Hasenfratz, "Prospects for perfect actions," *Nuclear Physics B*, vol. 63, no. 1–3, pp. 53–58, 1998.
- [35] W. Bietenholz, "Perfect scalars on the lattice," *International Journal of Modern Physics A*, vol. 15, no. 21, p. 3341, 2000.
- [36] B. Bahr and B. Dittrich, "Improved and perfect actions in discrete gravity," *Physical Review D*, vol. 80, no. 12, Article ID 124030, 2009.
- [37] B. Bahr and B. Dittrich, "Breaking and restoring of diffeomorphism symmetry in discrete gravity," in *Proceedings of the 25th Max Born Symposium on the Planck Scale*, pp. 10–17, July 2009.
- [38] B. Bahr, B. Dittrich, and S. Steinhaus, "Perfect discretization of reparametrization invariant path integrals," *Physical Review D*, vol. 83, no. 10, Article ID 105026, 2011.
- [39] B. Dittrich, "Diffeomorphism symmetry in quantum gravity models," <http://arxiv.org/abs/0810.3594>.



- [40] B. Bahr and B. Dittrich, “(Broken) Gauge symmetries and constraints in Regge calculus,” *Classical and Quantum Gravity*, vol. 26, no. 22, Article ID 225011, 2009.
- [41] B. Dittrich and P. A. Hoehn, “From covariant to canonical formulations of discrete gravity,” *Classical and Quantum Gravity*, vol. 27, no. 15, Article ID 155001, 2010.
- [42] M. Rocek and R. M. Williams, “Quantum Regge Calculus,” *Physics Letters B*, vol. 104, no. 1, pp. 31–37, 1981.
- [43] M. Rocek and R. M. Williams, “The Quantization Of Regge Calculus,” *Zeitschrift für Physik C*, vol. 21, no. 4, pp. 371–381, 1984.
- [44] P. A. Morse, “Approximate diffeomorphism invariance in near at simplicial geometries,” *Classical and Quantum Gravity*, vol. 9, no. 11, p. 2489, 1992.
- [45] H. W. Hamber and R. M. Williams, “Gauge invariance in simplicial gravity,” *Nuclear Physics B*, vol. 487, no. 1-2, pp. 345–408, 1997.
- [46] B. Dittrich, L. Freidel, and S. Speziale, “Linearized dynamics from the 4-simplex Regge action,” *Physical Review D*, vol. 76, no. 10, Article ID 104020, 2007.
- [47] T. Regge, “General relativity without coordinates,” *Il Nuovo Cimento*, vol. 19, no. 3, pp. 558–571, 1961.
- [48] T. Regge and R. M. Williams, “Discrete structures in gravity,” *Journal of Mathematical Physics*, vol. 41, no. 6, p. 3964, 2000.
- [49] L. Freidel and D. Louapre, “Diffeomorphisms and spin foam models,” *Nuclear Physics B*, vol. 662, no. 1-2, pp. 279–298, 2003.
- [50] G. T. Horowitz, “Exactly soluble diffeomorphism invariant theories,” *Communications in Mathematical Physics*, vol. 125, no. 3, pp. 417–437, 1989.
- [51] R. Dijkgraaf and E. Witten, “Topological gauge theories and group cohomology,” *Communications in Mathematical Physics*, vol. 129, no. 2, pp. 393–429, 1990.
- [52] M. de Wild Propitius and F. A. Bais, “Discrete gauge theories,” in *Banff 1994, Particles and Fields*, pp. 353–439, 1995.
- [53] M. Mackaay, “Finite groups, spherical 2-categories, and 4-manifold invariants,” *Advances in Mathematics*, vol. 153, no. 2, pp. 353–390, 2000.
- [54] A. Y. Kitaev, “Fault-tolerant quantum computation by anyons,” *Annals of Physics*, vol. 303, no. 1, pp. 2–30, 2003.
- [55] M. A. Levin and X.-G. Wen, “String-net condensation: a physical mechanism for topological phases,” *Physical Review B*, vol. 71, no. 4, Article ID 045110, 2005.
- [56] J. F. Plebański, “On the separation of Einsteinian substructures,” *Journal of Mathematical Physics*, vol. 18, no. 12, pp. 2511–2520, 1976.
- [57] J. W. Barrett and L. Crane, “Relativistic spin networks and quantum gravity,” *Journal of Mathematical Physics*, vol. 39, no. 6, Article ID 3296, 1998.
- [58] J. Engle, R. Pereira, and C. Rovelli, “The loop-quantum-gravity vertex-amplitude,” *Physical Review Letters*, vol. 99, no. 16, Article ID 161301, 2007.
- [59] J. Engle, E. Livine, R. Pereira, and C. Rovelli, “LQG vertex with finite Immirzi parameter,” *Nuclear Physics B*, vol. 799, no. 1-2, pp. 136–149, 2008.
- [60] E. R. Livine and S. Speziale, “A new spinfoam vertex for quantum gravity,” *Physical Review D*, vol. 76, no. 8, Article ID 084028, 2007.
- [61] L. Freidel and K. Krasnov, “A new spin foam model for 4d gravity,” *Classical and Quantum Gravity*, vol. 25, no. 12, Article ID 125018, 2008.
- [62] S. Alexandrov, “Simplicity and closure constraints in spin foam models of gravity,” *Physical Review D*, vol. 78, no. 4, Article ID 044033, 2008.
- [63] S. Alexandrov, “The new vertices and canonical quantization,” *Physical Review D*, vol. 82, no. 2, Article ID 024024, 2010.
- [64] B. Dittrich and J. P. Ryan, “Simplicity in simplicial phase space,” *Physical Review D*, vol. 82, Article ID 064026, 2010.
- [65] R. Oeckl, “Generalized lattice gauge theory, spin foams and state sum invariants,” *Journal of Geometry and Physics*, vol. 46, no. 3-4, pp. 308–354, 2003.
- [66] R. Oeckl, *Discrete Gauge Theory: From Lattices to Tqft*, Imperial College, London, UK, 2005.
- [67] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, H. Gomes, and F. Hellmann, “Asymptotic analysis of the EPRL four-simplex amplitude,” *Journal of Mathematical Physics*, vol. 50, no. 11, Article ID 112504, 2009.
- [68] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, H. Gomes, F. Hellmann, and R. Pereira, “Asymptotics of 4d spin foam models,” *General Relativity and Gravitation*, vol. 43, p. 2421, 2011.
- [69] F. Conrady and L. Freidel, “On the semiclassical limit of 4D spin foam models,” *Physical Review D*, vol. 78, no. 10, Article ID 104023, 2008.
- [70] S. Liberati, F. Girelli, and L. Sindoni, “Analogue models for emergent gravity,” <http://arxiv.org/abs/0909.3834>.
- [71] Z. C. Gu and X. G. Wen, “Emergence of helicity  $\pm 2$  modes (gravitons) from qubit models,” *Nuclear Physics B*, vol. 863, no. 1, pp. 90–129, 2012.
- [72] A. Hamma and F. Markopoulou, “Background independent condensed matter models for quantum gravity,” *New Journal of Physics*, vol. 13, Article ID 095006, 2011.
- [73] T. Fleming, M. Gross, and R. Renken, “Ising-Link quantum gravity,” *Physical Review D*, vol. 50, no. 12, pp. 7363–7371, 1994.
- [74] W. Beirl, H. Markum, and J. Riedler, “Two-dimensional lattice gravity as a spin system,” *International Journal of Modern Physics C*, vol. 5, p. 359, 1994.
- [75] E. Bittner, A. Hauke, H. Markum, J. Riedler, C. Holm, and W. Janke, “Z(2)-Regge versus standard Regge calculus in two dimensions,” *Physical Review D*, vol. 59, no. 12, Article ID 124018, 1999.
- [76] E. Bittner, W. Janke, and H. Markum, “Ising spins coupled to a four-dimensional discrete Regge skeleton,” *Physical Review D*, vol. 66, no. 2, Article ID 024008, 2002.
- [77] H. A. Kramers and G. H. Wannier, “Statistics of the two-dimensional ferromagnet. Part i,” *Physical Review*, vol. 60, no. 3, pp. 252–262, 1941.
- [78] R. Savit, “Duality in field theory and statistical systems,” *Reviews of Modern Physics*, vol. 52, no. 2, pp. 453–487, 1980.
- [79] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer, New York, NY, USA, 1991.
- [80] F. Girelli and H. Pfeiffer, “Higher gauge theory: differential versus integral formulation,” *Journal of Mathematical Physics*, vol. 45, no. 10, Article ID 3949, 2004.
- [81] F. Girelli, H. Pfeiffer, and E. M. Popescu, “Topological higher gauge theory: from BF to BFCG theory,” *Journal of Mathematical Physics*, vol. 49, no. 3, Article ID 032503, 2008.
- [82] J. Oitmaa, C. Hamer, and W. Zheng, *Series Expansion Methods for Strongly Interacting Lattice Models*, Cambridge University Press, Cambridge, UK, 2006.
- [83] F. Conrady, “Geometric spin foams, Yang-Mills theory and background-independent models,” <http://arxiv.org/abs/gr-qc/0504059>.



- [84] J. Drouffe and J. Zuber, "Strong coupling and mean field methods in lattice gauge theories," *Physics Reports*, vol. 102, no. 1-2, pp. 1-119, 1983.
- [85] M. Bojowald and A. Perez, "Spin foam quantization and anomalies," *General Relativity and Gravitation*, vol. 42, no. 4, pp. 877-907, 2010.
- [86] B. Bahr, "On knottings in the physical Hilbert space of LQG as given by the EPRL model," *Classical and Quantum Gravity*, vol. 28, no. 4, Article ID 045002, 2011.
- [87] B. Bahr, F. Hellmann, W. Kaminski, M. Kisielowski, and J. Lewandowski, "Operator spin foam models," *Classical and Quantum Gravity*, vol. 28, no. 10, Article ID 105003, 2011.
- [88] C. Rovelli and M. Smerlak, "In quantum gravity, summing is refining," *Classical and Quantum Gravity*, vol. 29, Article ID 055004, 2012.
- [89] G. De Las Cuevas, W. Dür, H. J. Briegel, and M. A. Martin-Delgado, "Mapping all classical spin models to a lattice gauge theory," *New Journal of Physics*, vol. 12, Article ID 043014, 2010.
- [90] J. B. Kogut, "An introduction to lattice gauge theory and spin systems," *Reviews of Modern Physics*, vol. 51, no. 4, pp. 659-713, 1979.
- [91] R. Oeckl and H. Pfeiffer, "The dual of pure non-Abelian lattice gauge theory as a spin foam model," *Nuclear Physics B*, vol. 598, no. 1-2, pp. 400-426, 2001.
- [92] F. Conrady, "Analytic derivation of dual gluons and monopoles from SU(2) lattice Yang-Mills theory. I. BF Yang-Mills representation," <http://arxiv.org/abs/hep-th/0610236>.
- [93] F. Conrady, "Analytic derivation of dual gluons and monopoles from SU(2) lattice Yang-Mills theory. II. Spin foam representation," <http://arxiv.org/abs/hep-th/0610237>.
- [94] M. Rakowski, "Topological modes in dual lattice models," *Physical Review D*, vol. 52, no. 1, pp. 354-357, 1995.
- [95] I. Montvay and G. Münster, *Quantum Fields on a Lattice*, Cambridge University Press, Cambridge, UK, 1994.
- [96] M. Martellini and M. Zeni, "Feynman rules and beta-function for the BF Yang-Mills theory," *Physics Letters B*, vol. 401, no. 1-2, pp. 62-68, 1997.
- [97] A. S. Cattaneo, P. Cotta-Ramusino, F. Fucito et al., "Four-dimensional Yang-Mills theory as a deformation of topological BF theory," *Communications in Mathematical Physics*, vol. 197, no. 3, pp. 571-621, 1998.
- [98] L. Smolin and A. Starodubtsev, "General relativity with a topological phase: an action principle," <http://arxiv.org/abs/hep-th/0311163>.
- [99] A. Starodubtsev, *Topological methods in quantum gravity [Ph.D. thesis]*, University of Waterloo, 2005.
- [100] D. Birmingham and M. Rakowski, "State sum models and simplicial cohomology," *Communications in Mathematical Physics*, vol. 173, no. 1, pp. 135-154, 1995.
- [101] D. Birmingham and M. Rakowski, "On Dijkgraaf-Witten type invariants," *Letters in Mathematical Physics*, vol. 37, no. 4, pp. 363-374, 1996.
- [102] S. W. Chung, M. Fukuma, and A. D. Shapere, "Structure of topological lattice field theories in three-dimensions," *International Journal of Modern Physics A*, vol. 9, no. 8, p. 1305, 1994.
- [103] M. Asano and S. Higuchi, "On three-dimensional topological field theories constructed from  $D^w(G)$  for finite groups," *Modern Physics Letters A*, vol. 9, no. 25, p. 2359, 1994.
- [104] J. C. Baez, "An introduction to spin foam models of BF theory and quantum gravity," in *Geometry and Quantum Physics*, vol. 543 of *Lecture Notes in Physics*, pp. 25-93, 2000.
- [105] V. Bonzom and M. Smerlak, "Bubble divergences from cellular cohomology," *Letters in Mathematical Physics*, vol. 93, no. 3, pp. 295-305, 2010.
- [106] V. Bonzom and M. Smerlak, "Bubble divergences from twisted cohomology," *Communications in Mathematical Physics*, vol. 312, no. 2, pp. 399-426, 2012.
- [107] B. Dittrich and J. P. Ryan, "Phase space descriptions for simplicial 4d geometries," *Classical and Quantum Gravity*, vol. 28, no. 6, Article ID 065006, 2011.
- [108] L. Freidel and K. Krasnov, "Spin foam models and the classical action principle," *Advances in Theoretical and Mathematical Physics*, vol. 2, pp. 1183-1247, 1999.
- [109] H. Pfeiffer, "Dual variables and a connection picture for the Euclidean Barrett-Crane model," *Classical and Quantum Gravity*, vol. 19, no. 6, pp. 1109-1137, 2002.
- [110] P. Cvitanovic, *Group Theory: Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, New Jersey, NJ, USA, 2008.
- [111] J. Kogut and L. Susskind, "Hamiltonian formulation of Wilson's lattice gauge theories," *Physical Review D*, vol. 11, no. 2, pp. 395-408, 1975.
- [112] J. Smit, *Introduction to Quantum Fields on a lattice: A Robust Mate*, vol. 15 of *Cambridge Lecture Notes in Physics*, Cambridge University Press, Cambridge, UK, 2002.
- [113] J. J. Halliwell and J. B. Hartle, "Wave functions constructed from an invariant sum over histories satisfy constraints," *Physical Review D*, vol. 43, no. 4, pp. 1170-1194, 1991.
- [114] C. Rovelli, "The projector on physical states in loop quantum gravity," *Physical Review D*, vol. 59, no. 10, Article ID 104015, 1999.
- [115] K. Noui and A. Perez, "Three-dimensional loop quantum gravity: physical scalar product and spin-foam models," *Classical and Quantum Gravity*, vol. 22, no. 9, pp. 1739-1761, 2005.
- [116] T. Thiemann, "Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity," *Physics Letters B*, vol. 380, no. 3-4, pp. 257-264, 1996.
- [117] T. Thiemann, "Quantum spin dynamics (QSD)," *Classical and Quantum Gravity*, vol. 15, no. 4, p. 839, 1998.
- [118] T. Thiemann, "QSD III: quantum constraint algebra and physical scalar product in quantum general relativity," *Classical and Quantum Gravity*, vol. 15, no. 5, pp. 1207-1247, 1998.
- [119] R. Sorkin, "Time evolution problem in Regge Calculus," *Physical Review D*, vol. 12, no. 2, pp. 385-396, 1975.
- [120] R. Sorkin, "Erratum: time-evolution problem in Regge calculus," *Physical Review D*, vol. 23, no. 2, p. 565, 1981.
- [121] J. W. Barrett, M. Galassi, W. A. Miller, R. D. Sorkin, P. A. Tuckey, and R. M. Williams, "A Parallellizable implicit evolution scheme for Regge calculus," *International Journal of Theoretical Physics*, vol. 36, no. 4, pp. 815-835, 1997.
- [122] B. Bahr, B. Dittrich, and S. He, "Coarse-graining free theories with gauge symmetries: the linearized case," *New Journal of Physics*, vol. 13, Article ID 045009, 2011.
- [123] T. Thiemann, "The Phoenix project: master constraint programme for loop quantum gravity," *Classical and Quantum Gravity*, vol. 23, no. 7, Article ID 2211, 2006.
- [124] B. Dittrich and T. Thiemann, "Testing the master constraint programme for loop quantum gravity: I. General framework," *Classical and Quantum Gravity*, vol. 23, no. 4, pp. 1025-1065, 2006.
- [125] B. Dittrich and T. Thiemann, "Testing the master constraint programme for loop quantum gravity. II: finite dimensional

- systems,” *Classical and Quantum Gravity*, vol. 23, no. 4, Article ID 1067, 2006.
- [126] D. Marolf, “Group averaging and refined algebraic quantization: where are we now?” <http://arxiv.org/abs/grqc/0011112>.
- [127] P. G. Bergmann, “Observables in general relativity,” *Reviews of Modern Physics*, vol. 33, no. 4, pp. 510–514, 1961.
- [128] C. Rovelli, “What is observable in classical and quantum gravity?” *Classical and Quantum Gravity*, vol. 8, no. 2, p. 297, 1991.
- [129] C. Rovelli, “Partial observables,” *Physical Review D*, vol. 65, Article ID 124013, 2002.
- [130] B. Dittrich, “Partial and complete observables for Hamiltonian constrained systems,” *General Relativity and Gravitation*, vol. 39, no. 11, pp. 1891–1927, 2007.
- [131] B. Dittrich, “Partial and complete observables for canonical general relativity,” *Classical and Quantum Gravity*, vol. 23, no. 22, Article ID 6155, 2006.
- [132] C. G. Torre, “Gravitational observables and local symmetries,” *Physical Review D*, vol. 48, no. 6, pp. 2373–2376, 1993.
- [133] S. Elitzur, “Impossibility of spontaneously breaking local symmetries,” *Physical Review D*, vol. 12, no. 12, pp. 3978–3982, 1975.
- [134] L. Freidel and D. Louapre, “Ponzano-Regge model revisited. I: Gauge fixing, observables and interacting spinning particles,” *Classical and Quantum Gravity*, vol. 21, no. 24, Article ID 5685, 2004.
- [135] K. Noui and A. Perez, “Three dimensional loop quantum gravity: coupling to point particles,” *Classical and Quantum Gravity*, vol. 22, no. 21, Article ID 4489, 2005.
- [136] K. Noui, “Three dimensional loop quantum gravity: particles and the quantum double,” *Journal of Mathematical Physics*, vol. 47, no. 10, Article ID 102501, 2006.
- [137] A. Perez, “On the regularization ambiguities in loop quantum gravity,” *Physical Review D*, vol. 73, Article ID 044007, 18 pages, 2006.
- [138] J. C. Baez and A. Perez, “Quantization of strings and branes coupled to BF theory,” *Advances in Theoretical and Mathematical Physics*, vol. 11, Article ID 060508, pp. 451–469, 2007.
- [139] W. J. Fairbairn and A. Perez, “Extended matter coupled to BF theory,” *Physical Review D*, vol. 78, no. 2, Article ID 024013, 2008.
- [140] W. J. Fairbairn, “On gravitational defects, particles and strings,” *Journal of High Energy Physics*, vol. 2008, no. 9, p. 126, 2008.
- [141] H. Bombin and M. A. Martin-Delgado, “Family of non-Abelian Kitaev models on a lattice: topological condensation and confinement,” *Physical Review B*, vol. 78, no. 11, Article ID 115421, 2008.
- [142] Z. Kadar, A. Marzuoli, and M. Rasetti, “Braiding and entanglement in spin networks: a combinatorial approach to topological phases,” *International Journal of Quantum Information*, vol. 7, p. 195, 2009.
- [143] Z. Kadar, A. Marzuoli, and M. Rasetti, “Microscopic description of 2d topological phases, duality, and 3D state sums,” *Advances in Mathematical Physics*, vol. 2010, Article ID 671039, 18 pages, 2010.
- [144] L. Freidel, J. Zapata, and M. Tylutki, *Observables in the Hamiltonian formulation of the Ponzano-Regge model [M.S. thesis]*, Uniwersytet Jagiellonski, Krakow, Poland, 2008.
- [145] H. Waelbroeck and J. A. Zapata, “Translation symmetry in 2+1 Regge calculus,” *Classical and Quantum Gravity*, vol. 10, no. 9, Article ID 1923, 1993.
- [146] H. Waelbroeck and J. A. Zapata, “2 +1 covariant lattice theory and t’Hooft’s formulation,” *Classical and Quantum Gravity*, vol. 13, p. 1761, 1996.
- [147] V. Bonzom, “Spin foam models and the Wheeler-DeWitt equation for the quantum 4-simplex,” *Physical Review D*, vol. 84, Article ID 024009, 13 pages, 2011.
- [148] A. Ashtekar, “New variables for classical and quantum gravity,” *Physical Review Letters*, vol. 57, no. 18, pp. 2244–2247, 1986.
- [149] A. Ashtekar, *New Perspectives in Canonical Gravity*, Monographs and Textbooks in Physical Science, Bibliopolis, Napoli, Italy, 1988.
- [150] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity*, World Scientific, Singapore, 1991.
- [151] M. Campiglia, C. Di Bartolo, R. Gambini, and J. Pullin, “Uniform discretizations: a new approach for the quantization of totally constrained systems,” *Physical Review D*, vol. 74, no. 12, Article ID 124012, 2006.
- [152] E. Alesci, K. Noui, and F. Sardelli, “Spin-foam models and the physical scalar product,” *Physical Review D*, vol. 78, no. 10, Article ID 104009, 2008.
- [153] E. Alesci and C. Rovelli, “A regularization of the hamiltonian constraint compatible with the spinfoam dynamics,” *Physical Review D*, vol. 82, no. 4, Article ID 044007, 2010.
- [154] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, “2D gravity and random matrices,” *Physics Report*, vol. 254, no. 1-2, pp. 1–133, 1995.
- [155] F. David, “Simplicial quantum gravity and random lattices,” <http://arxiv.org/abs/hep-th/9303127>.
- [156] F. David, “Planar diagrams, two-dimensional lattice gravity and surface models,” *Nuclear Physics B*, vol. 257, pp. 45–58, 1985.
- [157] V. A. Kazakov, “Bilocal regularization of models of random surfaces,” *Physics Letters B*, vol. 150, no. 4, pp. 282–284, 1985.
- [158] D. J. Gross and A. A. Migdal, “Nonperturbative two-dimensional quantum gravity,” *Physical Review Letters*, vol. 64, no. 2, pp. 127–130, 1990.
- [159] D. J. Gross and A. A. Migdal, “A nonperturbative treatment of two-dimensional quantum gravity,” *Nuclear Physics B*, vol. 340, no. 2-3, pp. 333–365, 1990.
- [160] V. A. Kazakov, “Ising model on a dynamical planar random lattice: exact solution,” *Physics Letters A*, vol. 119, no. 3, pp. 140–144, 1986.
- [161] D. V. Boulatov and V. A. Kazakov, “The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents,” *Physics Letters B*, vol. 186, no. 3-4, pp. 379–384, 1987.
- [162] V. A. Kazakov and A. A. Migdal, “Induced QCD at large  $N_c$ ,” *Nuclear Physics B*, vol. 397, no. 1-2, Article ID 920601, pp. 214–238, 1993.
- [163] P. H. Ginsparg and G. W. Moore, “Lectures on 2-D gravity and 2-D string,” <http://arxiv.org/abs/hep-th/9304011>.
- [164] P. Zinn-Justin and J. B. Zuber, “Matrix integrals and the counting of tangles and links,” <http://arxiv.org/abs/math-ph/9904019>.
- [165] J. L. Jacobsen and P. Zinn-Justin, “A Transfer matrix approach to the enumeration of knots,” <http://arxiv.org/abs/math-ph/0102015>.
- [166] P. Zinn-Justin, “The General  $O(n)$  quartic matrix model and its application to counting tangles and links,” *Communications in Mathematical Physics*, vol. 238, pp. 287–304, 2003.

- [167] P. Zinn-Justin and J. Zuber, "Matrix integrals and the generation and counting of virtual tangles and links," *Journal of Knot Theory and its Ramifications*, vol. 13, no. 3, pp. 325–355, 2004.
- [168] M. L. Mehta, *Random Matrices*, Academic Press, New York, NY, USA, 1991.
- [169] G. T. Hooft, "A planar diagram theory for strong interactions," *Nuclear Physics B*, vol. 72, no. 3, pp. 461–473, 1974.
- [170] G. T. Hooft, "A two-dimensional model for mesons," *Nuclear Physics B*, vol. 75, no. 3, pp. 461–470, 1974.
- [171] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, "Planar diagrams," *Communications in Mathematical Physics*, vol. 59, no. 1, pp. 35–51, 1978.
- [172] M. L. Mehta, "A method of integration over matrix variables," *Communications in Mathematical Physics*, vol. 79, no. 3, pp. 327–340, 1981.
- [173] C. Itzykson and J.-B. Zuber, "The planar approximation. II," *Journal of Mathematical Physics*, vol. 21, no. 3, pp. 411–421, 1979.
- [174] D. Bessis, C. Itzykson, and J. B. Zuber, "Quantum field theory techniques in graphical enumeration," *Advances in Applied Mathematics*, vol. 1, no. 2, pp. 109–157, 1980.
- [175] P. Di Francesco and C. Itzykson, "A Generating function for fatgraphs," *Annales de l'Institut Henri Poincaré (A) Physique Théorique*, vol. 59, pp. 117–140, 1993.
- [176] V. A. Kazakov, M. Staudacher, and T. Wynter, "Character expansion methods for matrix models of dually weighted graphs," *Communications in Mathematical Physics*, vol. 177, no. 2, pp. 451–468, 1996.
- [177] J. Ambjørn, D. V. Boulatov, A. Krzywicki, and S. Varsted, "The vacuum in three-dimensional simplicial quantum gravity," *Physics Letters B*, vol. 276, no. 4, pp. 432–436, 1992.
- [178] R. Gurau, "Colored group field theory," *Communications in Mathematical Physics*, vol. 304, pp. 69–93, 2011.
- [179] J. Ben Geloun, R. Gurau, and V. Rivasseau, "EPRL/FK group field theory," *Europhysics Letters*, vol. 92, no. 6, Article ID 60008, 2010.
- [180] R. Gurau, "Lost in translation: topological singularities in group field theory," *Classical and Quantum Gravity*, vol. 27, no. 23, Article ID 235023, 2010.
- [181] M. Smerlak, "Comment on 'Lost in translation: topological singularities in group field theory,'" *Classical and Quantum Gravity*, vol. 28, no. 17, Article ID 178001, 2011.
- [182] R. Gurau, "Reply to comment on 'Lost in translation: topological singularities in group field theory,'" *Classical and Quantum Gravity*, vol. 28, no. 17, Article ID 178002, 2011.
- [183] L. Freidel, R. Gurau, and D. Oriti, "Group field theory renormalization in the 3D case: power counting of divergences," *Physical Review D*, vol. 80, no. 4, Article ID 044007, 2009.
- [184] J. Magnen, K. Noui, V. Rivasseau, and M. Smerlak, "Scaling behavior of three-dimensional group field theory," *Classical and Quantum Gravity*, vol. 26, no. 18, Article ID 185012, 2009.
- [185] J. B. Geloun, T. Krajewski, J. Magnen, and V. Rivasseau, "Linearized group field theory and power-counting theorems," *Classical and Quantum Gravity*, vol. 27, no. 15, Article ID 155012, 2010.
- [186] R. Gurau, "The  $1/N$  Expansion of Colored Tensor Models," *Annales Henri Poincaré*, vol. 12, no. 5, pp. 829–847, 2011.
- [187] R. Gurau and V. Rivasseau, "The  $1/N$  expansion of colored tensor models in arbitrary dimension," *EPL*, vol. 95, no. 5, Article ID 50004, 2011.
- [188] R. Gurau, "The Complete  $1/N$  Expansion of Colored Tensor Models in Arbitrary Dimension," *Annales Henri Poincaré*, vol. 13, no. 3, pp. 399–423, 2012.
- [189] V. Bonzom, "New  $1/N$  expansions in random tensor models," *Journal of High Energy Physics*, vol. 1306, p. 062, 2013.
- [190] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, "Critical behavior of colored tensor models in the large  $N$  limit," *Nuclear Physics B*, vol. 853, no. 1, pp. 174–195, 2011.
- [191] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, "Critical behavior of colored tensor models in the large  $N$  limit," *Nuclear Physics B*, vol. 853, no. 1, pp. 174–195, 2011.
- [192] V. Bonzom, R. Gurau, and V. Rivasseau, "The Ising model on random lattices in arbitrary dimensions," *Physics Letters B*, vol. 711, no. 1, pp. 88–96, 2012.
- [193] V. Bonzom, "Multicritical tensor models and hard dimers on spherical random lattices," *Physics Letters A*, vol. 377, no. 7, pp. 501–506, 2013.
- [194] V. Bonzom and H. Erbin, "Coupling of hard dimers to dynamical lattices via random tensors," *Journal of Statistical Mechanics*, vol. 1209, Article ID P09009, 2012.
- [195] D. Benedetti and R. Gurau, "Phase transition in dually weighted colored tensor models," *Nuclear Physics B*, vol. 855, no. 2, pp. 420–437, 2012.
- [196] J. Ambjørn, B. Durhuus, and T. Jónsson, "Summing over all genera for  $d > 1$ : a toy model," *Physics Letters B*, vol. 244, no. 3–4, pp. 403–412, 1990.
- [197] P. Bialas and Z. Burda, "Phase transition in uctuating branched geometry," *Physics Letters B*, vol. 384, no. 1–4, pp. 75–80, 1996.
- [198] R. Gurau and J. P. Ryan, "Melons are branched polymers," <http://arxiv.org/abs/1302.4386>.
- [199] R. Gurau, "Universality for random tensors," <http://arxiv.org/abs/1111.0519>.
- [200] R. Gurau, "A generalization of the Virasoro algebra to arbitrary dimensions," *Nuclear Physics B*, vol. 852, no. 3, pp. 592–614, 2011.
- [201] R. Gurau, "The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders," *Nuclear Physics B*, vol. 865, no. 1, pp. 133–147, 2012.
- [202] V. Bonzom, "Revisiting random tensor models at large  $N$  via the Schwinger-Dyson equations," *Journal of High Energy Physics*, vol. 1303, p. 160, 2013.
- [203] R. De Pietri and C. Petronio, "Feynman diagrams of generalized matrix models and the associated manifolds in dimension four," *Journal of Mathematical Physics*, vol. 41, no. 10, pp. 6671–6688, 2000.
- [204] D. V. Boulatov, "A Model of three-dimensional lattice gravity," *Modern Physics Letters A*, vol. 7, no. 18, p. 1629, 1992.
- [205] H. Ooguri, "Topological lattice models in four-dimensions," *Modern Physics Letters A*, vol. 7, no. 30, pp. 2799–2810, 1992.
- [206] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli, "Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space," *Nuclear Physics B*, vol. 574, no. 3, pp. 785–806, 2000.
- [207] A. Baratin, F. Girelli, and D. Oriti, "Diffeomorphisms in group field theories," *Physical Review D*, vol. 83, no. 10, Article ID 104051, 2011.
- [208] J. Ben Geloun and V. Bonzom, "Radiative corrections in the Boulatov-Ooguri tensor model: the 2-point function," *International Journal of Theoretical Physics*, vol. 50, no. 9, pp. 2819–2841, 2011.



- [209] J. B. Geloun and V. Rivasseau, “A renormalizable 4-dimensional tensor field theory,” *Communications in Mathematical Physics*, vol. 318, no. 1, pp. 69–109, 2013.
- [210] J. B. Geloun and D. O. Samary, “3D tensor field theory: renormalization and one-loop  $\beta$ -functions,” <http://arxiv.org/abs/1201.0176>.
- [211] J. B. Geloun, “Two and four-loop  $\beta$ -functions of rank 4 renormalizable tensor field theories,” *Classical and Quantum Gravity*, vol. 29, Article ID 235011, 2012.
- [212] J. B. Geloun and V. Rivasseau, “Addendum to ‘A renormalizable 4-dimensional tensor field theory,’” <http://arxiv.org/abs/1209.4606>.
- [213] J. B. Geloun, “Asymptotic freedom of rank 4 tensor group field theory,” <http://arxiv.org/abs/1210.5490>.
- [214] J. B. Geloun and E. R. Livine, “Some classes of renormalizable tensor models,” <http://arxiv.org/abs/1207.0416>.
- [215] S. Carrozza and D. Oriti, “Bounding bubbles: the vertex representation of 3d group field theory and the suppression of pseudomanifolds,” *Physical Review D*, vol. 85, no. 4, Article ID 044004, 2012.
- [216] S. Carrozza and D. Oriti, “Bubbles and jackets: new scaling bounds in topological group field theories,” *Journal of High Energy Physics*, vol. 1206, p. 092, 2012.
- [217] S. Carrozza, D. Oriti, and V. Rivasseau, “Renormalization of tensorial group field theories: Abelian  $U(1)$  models in four dimensions,” <http://arxiv.org/abs/1207.6734>.
- [218] S. Carrozza, D. Oriti, and V. Rivasseau, “Renormalization of an  $SU(2)$  tensorial group field theory in three dimensions,” <http://arxiv.org/abs/1303.6772>.
- [219] D. Oriti and L. Sindoni, “Toward classical geometrodynamics from the group field theory hydrodynamics,” *New Journal of Physics*, vol. 13, Article ID 025006, 2011.
- [220] L. Freidel, D. Oriti, and J. Ryan, “A group field theory for 3d quantum gravity coupled to a scalar field,” <http://arxiv.org/abs/gr-qc/0506067>.
- [221] D. Oriti and J. Ryan, “Group field theory formulation of 3-D quantum gravity coupled to matter fields,” *Classical and Quantum Gravity*, vol. 23, pp. 6543–6576, 2006.
- [222] R. J. Dowdall, “Wilson loops, geometric operators and fermions in 3d group field theory,” <http://arxiv.org/abs/0911.2391>.
- [223] F. Girelli and E. R. Livine, “A deformed Poincaré invariance for group field theories,” *Classical and Quantum Gravity*, vol. 27, no. 24, Article ID 245018, 2010.
- [224] M. Dupuis, F. Girelli, and E. R. Livine, “Spinors group field theory and Voros star product: first contact,” *Physical Review D*, vol. 86, Article ID 105034, 2012.
- [225] W. J. Fairbairn and E. R. Livine, “3D spinfoam quantum gravity: matter as a phase of the group field theory,” *Classical and Quantum Gravity*, vol. 24, no. 20, pp. 5277–5297, 2007.
- [226] F. Girelli, E. R. Livine, and D. Oriti, “4D deformed special relativity from group field theories,” *Physical Review D*, vol. 81, no. 2, Article ID 024015, 2010.
- [227] E. R. Livine, D. Oriti, and J. P. Ryan, “Effective Hamiltonian constraint from group field theory,” *Classical and Quantum Gravity*, vol. 28, no. 24, Article ID 245010, 2011.
- [228] J. B. Geloun, “Classical group field theory,” *Journal of Mathematical Physics*, Article ID 022901, p. 53, 2012.
- [229] G. Calcagni, S. Gielen, and D. Oriti, “Group field cosmology: a cosmological field theory of quantum geometry,” *Classical and Quantum Gravity*, vol. 29, no. 10, Article ID 105005, 2012.
- [230] V. Bonzom, R. Gurau, and V. Rivasseau, “Random tensor models in the large  $N$  limit: uncoloring the colored tensor models,” *Physical Review D*, vol. 85, no. 8, Article ID 084037, 2012.
- [231] V. A. Kazakov, “The appearance of matter fields from quantum fluctuations of 2D-gravity,” *Modern Physics Letters A*, vol. 4, p. 2125, 1989.
- [232] V. A. Kazakov, “Exactly solvable Potts models, bond- and tree-like percolation on dynamical (random) planar lattice,” *Nuclear Physics B*, vol. 4, pp. 93–97, 1988.
- [233] V. Bonzom and M. Smerlak, “Bubble divergences from cellular cohomology,” *Letters in Mathematical Physics*, vol. 93, no. 3, pp. 295–305, 2010.
- [234] V. Bonzom and M. Smerlak, “Bubble Divergences from Twisted Cohomology,” *Communications in Mathematical Physics*, pp. 1–28, 2012.
- [235] V. Bonzom and M. Smerlak, “Bubble divergences: sorting out topology from cell structure,” *Annales Henri Poincaré*, vol. 13, no. 1, pp. 185–208, 2012.
- [236] F. Markopoulou, “An algebraic approach to coarse graining,” <http://arxiv.org/abs/hep-th/0006199>.
- [237] F. Markopoulou, “Coarse graining in spin foam models,” *Classical and Quantum Gravity*, vol. 20, no. 5, pp. 777–799, 2003.
- [238] R. Oeckl, “Renormalization of discrete models without background,” *Nuclear Physics B*, vol. 657, no. 1–3, pp. 107–138, 2003.
- [239] M. Levin and C. P. Nave, “Tensor renormalization group approach to two-dimensional classical lattice models,” *Physical Review Letters*, vol. 99, no. 12, Article ID 120601, 2007.
- [240] Z.-C. Gu, M. Levin, and X.-G. Wen, “Tensor-entanglement renormalization group approach as a unified method for symmetry breaking and topological phase transitions,” *Physical Review B*, vol. 78, no. 20, Article ID 205116, 2008.
- [241] L. Tagliacozzo and G. Vidal, “Entanglement renormalization and gauge symmetry,” *Physical Review B*, vol. 83, no. 11, Article ID 115127, 2011.
- [242] B. Dittrich, F. C. Eckert, and M. Martin-Benito, “Coarse graining methods for spin net and spin foam models,” *New Journal of Physics*, vol. 14, Article ID 35008, 2012.
- [243] B. Dittrich, “From the discrete to the continuous: towards a cylindrically consistent dynamics,” *New Journal of Physics*, vol. 14, Article ID 123004, 2012.
- [244] L. Freidel and E. R. Livine, “Effective 3D quantum gravity and effective noncommutative quantum field theory,” *Physical Review Letters*, vol. 96, no. 22, Article ID 221301, 2006.
- [245] L. Freidel and S. Majid, “Noncommutative harmonic analysis, sampling theory and the Duflo map in 2+1 quantum gravity,” *Classical and Quantum Gravity*, vol. 25, no. 4, Article ID 045006, 2008.
- [246] A. Baratin, B. Dittrich, D. Oriti, and J. Tambornino, “Non-commutative flux representation for loop quantum gravity,” *Classical and Quantum Gravity*, vol. 28, no. 17, Article ID 175011, 2011.